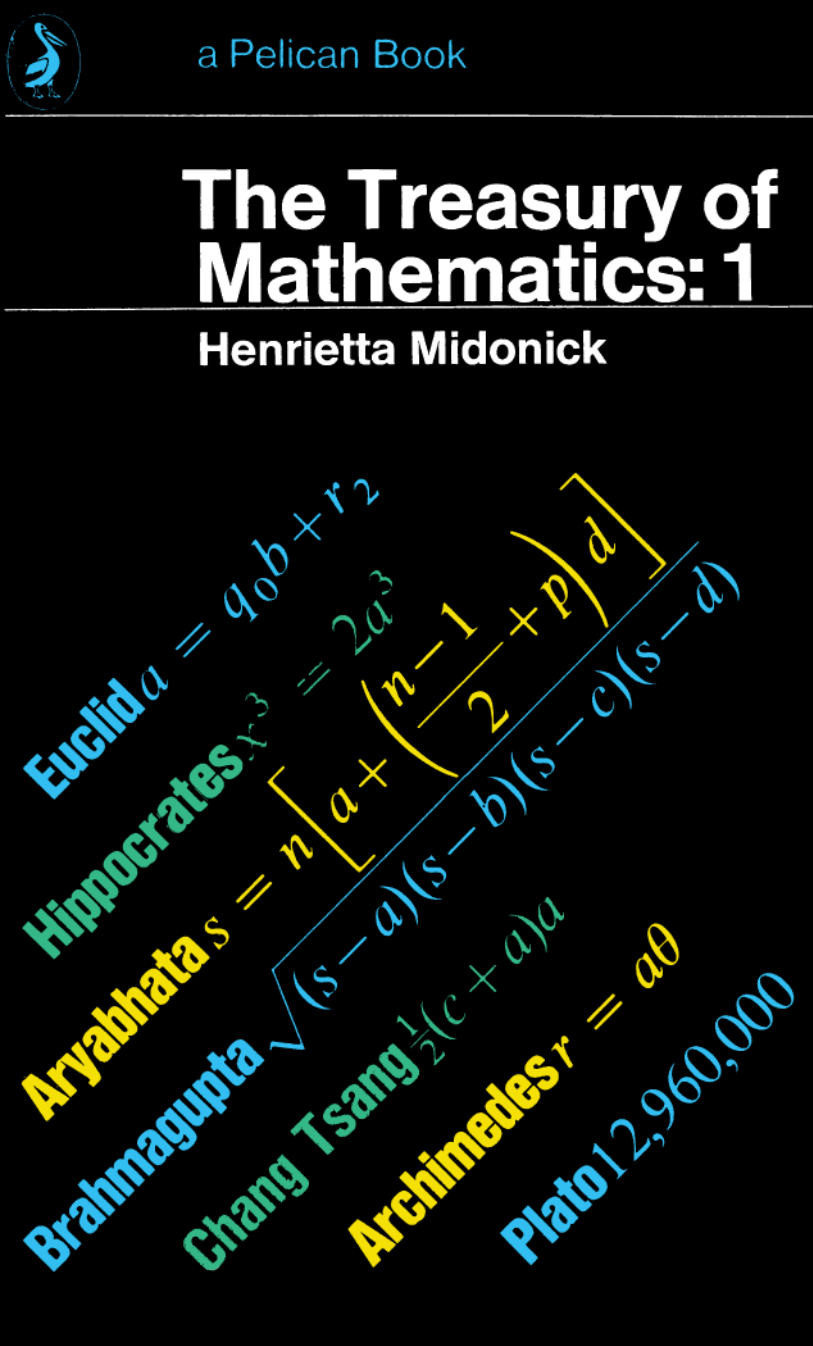
a Pelican Book

The Treasury of Mathematics: 1

Henrietta Midonick


Euclid $a = q_0 b + r_2$
Hippocrates $x^3 = 2a^3$
Aryabhata $s = n\left[a + \left(\frac{n-1}{2} + p\right) d\right]$
Brahmagupta $\sqrt{(s-a)(s-b)(s-c)(s-d)}$
Chang Tsang $\frac{1}{2}(c+a)a$
Archimedes $r = a\theta$
Plato $12{,}960{,}000$

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Mathematics is the only true international language. Men can communicate more directly, precisely and logically in pure mathematics than in any other tongue. Moreover we have much to learn from the achievements of past civilizations in this field: even modern computers have not fathomed all the intricacies of Stonehenge. In this fascinating collection of original sources (many of them published in a popular edition for the first time) Henrietta Midonick shows individual mathematicians grappling with varied problems – some practical, such as architecture, money valuation, mechanics, astronomy and calendar calculation; others verging on philosophy, such as the existence of zero and the concept of infinity. Her arrangement also demonstrates the growth of key ideas in geometry, arithmetic, logic and calculus.

Volume I documents the growth of the mathematical science in the civilizations of Babylon, Ancient Egypt, the Mayas, India and China, and assesses the revolutionary discoveries of Plato, Archimedes and Euclid in classical antiquity.

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THE TREASURY OF MATHEMATICS

VOLUME 1

THE TREASURY OF

MATHEMATICS

VOLUME ONE

A COLLECTION OF
SOURCE MATERIAL IN MATHEMATICS
EDITED AND PRESENTED WITH
INTRODUCTORY BIOGRAPHICAL AND
HISTORICAL SKETCHES BY

HENRIETTA MIDONICK

Revised by
Minetta and Reginald Vesselo

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FOREWORD

IN these two volumes, I have collected significant contributions to the development of mathematics, selecting such as changed and influenced the course of human action, paying special attention to those whose lasting effect is discernible today. The form adopted for this publication is one which is, I believe, excellently suited to the task of making the landmarks of mathematical advance live again. Here we shall have not mere narration, nor simple paraphrasing, but the original master-words of the greatest inventive minds of the world, rendered easily accessible to student and layman.

There can be no easy or quick solution to the basic problem of selection in the preparation of such a work. The ablest and most profound mathematicians frequently prove to be prolific writers. Five volumes are required for the *Dialogues of Plato*, three for Euclid's *Elements* (the most famous of nine or ten works attributed to him). The collected papers of Charles S. Peirce appear in six volumes, those of James J. Sylvester in four, those of Arthur Cayley in thirteen. Moreover, for most mathematicians complete collections have never been issued. Mathematical works containing vital, original contributions are, in general, widely scattered in books and professional periodicals over many lands. The essential part of the task of selection reduces to ascertaining where and how, in the prodigious, vastly extended outpouring of the human mind, where and how to seize a high moment of inspiration and inventiveness, a moment of passionate enthusiasm and vigorous creativeness, a moment whose fire for ever fascinates.

The matter of translation in connexion with an item to be included in the collection cannot by any means be regarded as one of secondary importance. Readers in English-speaking

countries, apart from a comparatively small number of persons sufficiently trained in foreign languages, are dependent upon translations for a full view of what the world has accomplished in mathematics. Even in the case of English writers, the practice of writing scholarly works in Latin persisted for a long time. The works of Isaac Barrow published in the second half of the seventeenth century were in Latin. Newton's *Principia* was first translated into English from the Latin in 1729. Fortunately we possess superb translations of many works composed by the ancient Greek masters. The mysteries of mathematical inscriptions on clay, stone and papyrus are now dispelled by the combined forces of modern archaeology, linguistics and mathematical science. Leading us step by step through the meanings of primitive symbols they join to transport us back in time to the great civilizations that flourished before Greece, there to observe examples of mathematics set down by ancient hands. In a translation, clearly, the search must be for one which not only reflects the author's thought and style, but also tends, through the subtleties of language, to minimize distance both in time and space.

Consideration along these lines entered basically into the choice of articles collected in the *Treasury of Mathematics*. The selections represent pivotal points in the advance and spread of mathematical knowledge from its earliest appearance to the rise of the fundamental notions of the calculus, mathematical philosophy and logic. To the justly famous treasures of Europe have been added valuable finds of recent discovery. These together with the mathematical works arising in the Far East, the Middle East and the Americas are dramatic evidence of the world-embracing grandeur of the progress of mathematics. Wherever possible, a selection is given under the name of the great innovator to whom we are indebted for it. Mathematics in many aspects will be found here: the exposition of the concept of number by Gottlob Frege, the so-called 'Chinese remainder problem' first stated by Sun Tsu, the revelation by Archimedes of his way of thinking, the eloquent and effective plea by Simon Stevin for the adoption of the decimal system of notation, the shaded diagrams introduced in symbolic logic by John Venn, the algebra of Omar Khayyám, and so on. Where the evidences of mathematical knowledge and

skill have lost all vestiges of individual contributions, source material has been drawn from the realm of the specialist's bookshelf. We turn, for example, to T. E. Peet for the analysis of Egyptian arithmetic and mensuration found in the Moscow Mathematical Papyrus, to C. P. Bowditch and S. G. Morley for an explanation of Mayan numeration and the amazingly accurate Mayan calendrical system, to L. L. Locke for a lucid account of the Peruvian *quipu*, to O. Neugebauer and A. Sachs for the results of recent researches on Babylonian mathematics. Occasion has been taken, too, to include the reproduction of some pages from rare old editions so that, to the unhurried reader, the flavour of an old work and the personality of its author may be the more fully manifest.

Each of the fifty-four articles is preceded by a brief introductory essay in which I have sketched relevant biographical and historical details. The essays serve the purpose of general orientation. They also serve to indicate that the selections, despite their wide variety, do not stand unrelated to one another or to world history, but are indeed prime instances of the continuity of the intellectual growth of mankind, and of the reality of the interplay of mathematical ideas with the economic, political and religious forces that shape men's lives.

HENRIETTA O. MIDONICK

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ARCHIMEDES

Selections from *On Spirals*; *The Method*; *The Quadrature of the Parabola*, all from *The Works of Archimedes*, ed. T. L. Heath (New York: Dover).

ĀRYABHATA, THE ELDER

Selections from *The Āryabhatīya of Āryabhata*, translated by Walter Eugene Clark (Chicago: University of Chicago Press, 1930).

THE BABYLONIANS

Selections from: 1. *The Exact Sciences in Antiquity*, Otto Neugebauer (Princeton: Princeton University Press, 1952); 2. *Mathematical Cuneiform Text*, eds. Otto Neugebauer and A. Sachs (New Haven: American Oriental Society, American Schools of Oriental Research, 1945).

THE BAKHSHALI MANUSCRIPT

Selections from A. F. Rudolf Hoernle, 'The Bakhshali Manuscript', *The Indian Antiquary*, vol. XVII, 1888.

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BRAHMAGUPTA

Selections from the *Brahma-sphuta-sidd'hanta* in *Algebra with Arithmetic and Mensuration*, translated by Henry Thomas Colebrooke (London: John Murray, 1817).

CHANG TSANG

Selections from: 1. *Chinese Researches*, Alexander Wylie (Shanghai, 1897); 2. *The Development of Mathematics in China and Japan*, Yoshio Mikami, in *Abhandlungen zur Geschichte der Mathematischen Wissenschaften* (Leipzig, 1913).

CH'IN CHIU-SHAO

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CHOU KUNG

Selections from *Chou-Pi* in *Chinese Researches*, Alexander Wylie (Shanghai, 1897).

CHU CHI-CHIEH

Selections from: 1. *Suan-hsiao Chi-meng* in *A History of Japanese Mathematics*, D. E. Smith and Yoshio Mikami (Chicago: Open Court, 1914); 2. *Precious Mirror of the Four Origins* in *Chinese Researches*, Alexander Wylie (Shanghai, 1897), and in *The Development of Mathematics in China and Japan*, Yoshio Mikami, in *Abhandlungen zur Geschichte der Mathematischen Wissenschaften* (Leipzig, 1913).

EUCLID OF ALEXANDRIA

Selections from the *Elements* in *The Thirteen Books of Euclid's Elements*, translated by Thomas Heath, 2nd edn (New York: Dover, 1956).

HIPPOCRATES OF CHIOS

Selections from Eudemus's *History of Geometry* in *Greek Geometry from Thales to Euclid*, ed. George J. Allman (London: Longmans, Green, 1889).

HSIA-HOU YANG

Selections from Louis Vanhee, 'Arithmetical Classic of Hsia-hou Yang', *American Mathematical Monthly*, vol. XXXI, 1924.

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Selections from *The Algebra of Mohammed ben Musa*, translated by Frederic Rosen (London, 1831).

LI YEH

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LIU HUI

Extracts from *Sea Island Arithmetic Classic* in *The Development of Mathematics in China and Japan*, Yoshio Mikami, in *Abhandlungen zur Geschichte der Mathematischen Wissenschaften* (Leipzig, 1913).

THE MAYA CIVILIZATION

Selections from: 1. *The Numeration, Calendar Systems and Astronomical Knowledge of the Mayas*, Charles P. Bowditch (Cambridge: C.U.P., 1910); 2. *An Introduction to the Study of the Maya Hieroglyphs*, Sylvanus Morley (Washington: Printing Office, 1915); 3. *The Ancient Maya*, Sylvanus Morley (Stanford: Stanford University Press, 1947).

THE MOSCOW PAPYRUS

Selections from 'Four Geometrical Problems from the Moscow Mathematical Papyrus', Battiscombe Gunn and T. Eric Peet, *Journal of Egyptian Archaeology*, vol. XV, parts III and IV, 1929.

R. NEHEMIAH

Selections from *The Mishnat ha-Middot* in *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik* (Berlin: Springer, 1932), translated into English by Solomon Gandz.

PAPPUS

Selections from *The Mathematical Collection*, translated by Ivor Thomas in *Greek Mathematical Works*, vol. II (Cambridge, Mass.: Harvard University Press, 1941).

THE PERUVIAN QUIPU

Selections from *The Ancient Quipu or Peruvian Knot Record*, Leslie Leland Locke (American Museum of Natural History, 1923).

PLATO

Selections from *The Dialogues of Plato*, translated by B. Jowett, 2nd edn (Oxford: Clarendon Press, 1875).

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PROCLUS

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THE RHIND MATHEMATICAL PAPYRUS

Selections from: 1. *The Rhind Mathematical Papyrus*, T. Eric Peet (London: Hodder and Stoughton, 1923); 2. *The Rhind Mathematical Papyrus*, Arnold B. Chace and H. P. Manning, vols. I and II (Oberlin: Mathematical Association of America, 1927).

SUN-TSU

Selections from *Sun-Tsu Suan-ching* in *The Development of Mathematics in China and Japan*, Yoshio Mikami, in *Abhandlungen zur Geschichte der Mathematischen Wissenschaften* (Leipzig, 1913).

WAN WANG

Selections from *Yi King*, translated by James Legge in *The Sacred Books of the East*, vol. XVI (Oxford: Clarendon Press, 1882).

WANG HS'IAO-T'UNG

Selections from the *Ch'i-ku Suan-ching* in *The Development of Mathematics in China and Japan*, Yoshio Mikami, in *Abhandlungen zur Geschichte der Mathematischen Wissenschaften* (Leipzig, 1913).

YANG HUI

Selections from *Analysis of the Nine Sections* in *The Development of Mathematics in China and Japan*, Yoshio Mikami, in *Abhandlungen zur Geschichte der Mathematischen Wissenschaften* (Leipzig, 1913).

THE BEGINNINGS

I

THE BABYLONIANS

(3rd Millennium B.C. to 0)

THE startling finds of a highly developed ancient culture in the Mesopotamian Valley, made by French and British archaeologists in the decade of the 1840s, were followed in the next hundred years by scientific exploration of the ruins of the ancient cities of Babylonia. Archaeological excavation of the mounds marking the ancient sites revealed a complex civilization which flourished more than four thousand years ago in the fertile crescent of land watered by the Tigris and Euphrates Rivers. It was a civilization characterized by prospering agricultural settlements. Networks of canals, whose purpose was to reclaim swamps and feed parched areas, were found. Here, according to Herodotus, wheat returned to the sower from two hundred- to three hundredfold. There were also numerous individuals who were occupied with laws, religion, science, art, architecture, trading, teaching and engineering. Large palace sculptures, metal bas-reliefs, copper and bronze figures, painted pottery, cylinder seals, and other artifacts of great antiquity were recovered. Most significant of all the finds, however, were the thousands upon thousands of clay tablets unearthed at Nineveh, Assur, Nippur and other sites, bearing the written records of the economic, juridical, educational and scientific phases of daily life in this remote period previously known to us, in the main, through Biblical allusions. The cuneiform writings incised on the tablets, painstakingly copied and deciphered by scholars of our own time, disclose an intricate system of trading in which weights and measures were standardized, and exchange was in kind or in money. Gold, silver and copper, used for ornamentation, also served as a medium of exchange. The shekel and fractional parts of the shekel were in common use. Skilled in the working of clay and metals, the Babylonians brought the carving of precious stones to a fine art. Great numbers of the tablets served as legal documents relating to business contracts, loans, property, inheritance and marriage. Astronomical observations formed the basis of a calendar, the initial motivation being principally the timing of religious festivals.

Later, under the intense interest attaching to all methods of divination, the vogue for astrological interpretations of astronomy grew to great proportions.

Underlying the commercial, monetary and astronomical systems employed by the Babylonians, were their achievements in mathematics, made known to us through the continuing study and decipherment of the cuneiform inscriptions on the ancient tablets by scholars who happily combine the requisite linguistic power with the technical scientific knowledge needed. Although the tablets containing mathematical subject matter are only a small portion of the huge collections of tablets recovered, they are cogent evidence of a high degree of skill and originality on the part of Babylonian mathematicians. Outstanding among the contributions of the Babylonians to the development of mathematics is their use of place values in their system of numeration. The remarkable accuracy of Babylonian astronomical computations was due in no small measure to the convenient form of their number system. The first record we possess of any study of problems in number theory is contained in a Babylonian mathematical tablet (Plimpton 322, Columbia University, N.Y.).

Most of the mathematical tablets have been classified either as 'table texts' or as 'problem texts'. The latter appear in many cases to have been school texts illustrating rules for the solution of problems. The table texts were ever-present aids both for instructional problems and for practical use. They included tables of reciprocals, squares and square roots, cubes and cube roots as well as multiplication tables. With these tables at their disposal for the numerical calculations involved, the Babylonians developed many ingenious rules and methods for the solution of a wide variety of mathematical problems.

THE EXACT SCIENCES IN ANTIQUITY

by O. Neugebauer

CHAPTER I

The texts of which I speak are clay tablets, generally about the size of a hand, inscribed with signs which were pressed into the surface of the once soft clay by means of a sharpened stylus. This script is called 'cuneiform', i.e. wedge-shaped, because the individual impressions have a deeper 'head' and a finer line at the

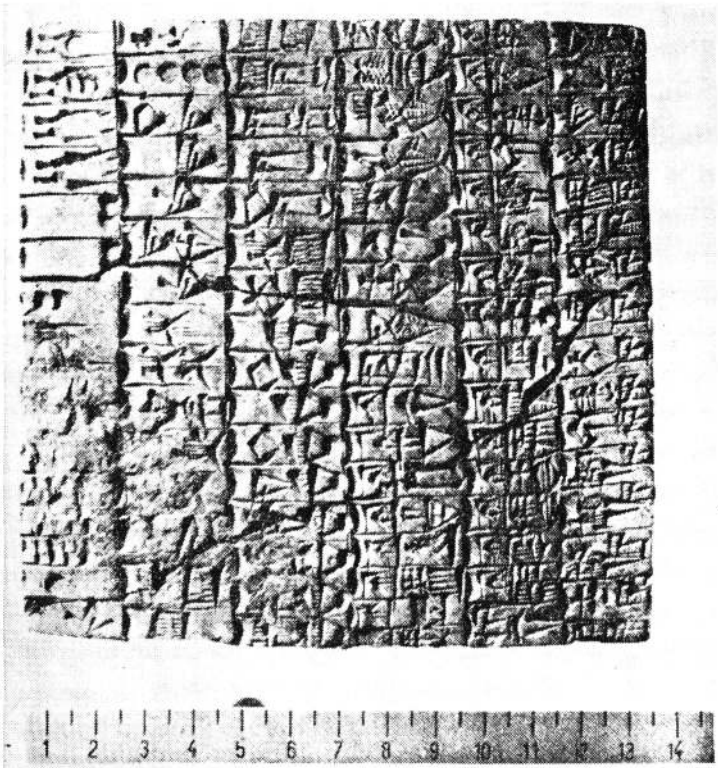


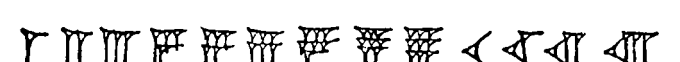
Plate I. Economic text from Fara.

end, thus resembling a wedge. Cuneiform tablets with mathematical contents are known to us mostly from the so-called 'Old-Babylonian' period, about 1600 B.C. No astronomical texts of any scientific significance exist from this period, while the mathematical texts already show the highest level ever attained in Babylonia.

The second period from which we have a larger number of texts is the latest period of Babylonian history, when Mesopotamia had become a part of the empire of Alexander's successors, the 'Seleucids'. This period, from about 300 B.C. to the beginning of our era, has furnished us with a great number of astronomical texts of a most remarkable mathematical character, fully comparable to the astronomy of the Almagest. Mathematical texts from this period are scarce, but they suffice nevertheless to demonstrate that the knowledge of Old-Babylonian mathematics had not been lost during the intervening 1,300 years for which texts are lacking.

12. The development of the numerical notations in Mesopotamia took as many centuries as the development of writing from a crude picture script to a well defined system of complicated signs. We shall for the moment deal only with the final product as it appears in the mathematical texts of the Old-Babylonian period. And we shall again use the most direct approach by deciphering an actual text.

Plate 2 shows a tablet whose size is about $3\frac{1}{8}$ by 2 inches (and about $\frac{3}{4}$ of an inch thick). In the middle of the text is visible a column of signs which obviously represent numbers in ascending order. The tablet is not quite cleaned from incrustation of salt or dirt but it is clear that the signs look about as follows:



Counting of the vertical wedges leads directly to the readings 1, 2, 3, etc. up to 9. Then follows which must be 10, and consequently we can also read the remaining signs as 11, 12, 13. Using this exceedingly plausible hypothesis, we should also be

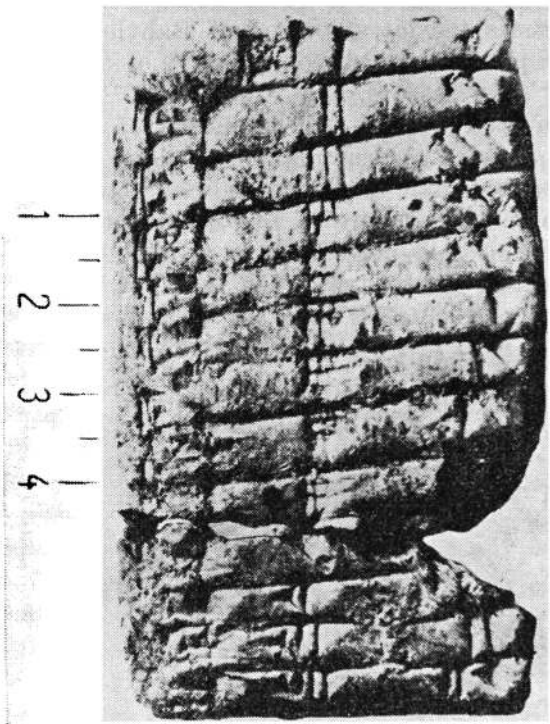


Plate 2. Multiplication Table for 10.

able to read the right-hand column of signs. The first five look as follows:

Obviously we must read these signs as 10, 20, 30, 40, 50 if the first sign represents 10 as we have established in our first list. But what follows is

which we transcribe consistently as

1 1,10 1,20 * * * 2 2,10

each * indicating a broken line. These signs continue the previous ones if we interpret the first '1' as 60 and then read 1,10 as $60+10=70$ and 1,20 as $60+20=80$. The broken lines should contain 90, 100 and 110. The next sign '2' should be 120, in excellent agreement with our interpretation of '1' as 60, while the last sign 2,10 must be $120+10=130$. Thus we have obtained all multiples of 10 from 10 to 130, line by line, corresponding to the numbers 1 to 13. In other words, our table is a multiplication table for 10, which we now can transcribe as follows:

1	10	8	1,20
2	20	9	1,30
3	30	10	1,40
4	40	11	1,50
5	50	12	2
6	1	13	2,10
7	1,10		

The notation $1,10=70$ $1,20=80$ $2,10=130$ etc. is 'sexagesimal' in the sense that 60 units of one kind are written as 1 of the next higher order.

13. The example of our present system of numeration for degrees, hours, measures and ordinary numbers should suffice totally to discredit the popular idea that a number system was 'invented' at a certain moment. Yet innumerable 'reasons' have been advanced why the Babylonians used the basis 60 for their number system. I shall not make any attempt to discuss here the history of the sexagesimal system in any detail, but a few points must be mentioned because they are of importance for the historical approach to the development of number systems as a whole.

First of all, there exists a common misconception as to the generality of the use of the sexagesimal system. The very same tablet which contains hundreds of sexagesimal numbers, column beside column, to compute the dates of the new moons for a

given year, might end with a 'colophon' containing the name of the owner of the tablet, the name of the scribe, and the date of writing of the text, the year being expressed in the form 2 *me* 25 '2 hundred 25' where the main text would express the very same date sexagesimally as 3,45. In other words, it is only in strictly mathematical or astronomical contexts that the sexagesimal system is consistently applied. In all other matters (dates, measures of weight, areas, etc.), use was made of mixed systems which have their exact parallel in the chaos of 60-division, 24-division, 12-division, 10-division, 2-division which characterizes the units of our own civilization. The question of the origin of the sexagesimal system is therefore inextricably related to the much more complex problem of the history of many concurrent numerical notations and their innumerable local and chronological variations.

14. The Babylonian place value notation shows in its earlier development two disadvantages which are due to the lack of a symbol for zero. The first difficulty consists in the possibility of misreading a number 1 20 as 1,20 = 80 when actually 1,0,20 = 3620 was meant. Occasionally this ambiguity is overcome by separating the two numbers very clearly if a whole sexagesimal place is missing. But this method is by no means strictly applied and we have many cases where numbers are spaced widely apart without any significance. In the latest period, however, when astronomical texts were computed, a special symbol for 'zero' was used. This symbol also occurs earlier as a separation mark between sentences, and I therefore transcribe it by a 'period'. Thus we find in Seleucid astronomical texts many instances of numbers like 1,.,20 or even 1,.,.,20 which apply exactly the same principle as our 201 or 2001.

But even in the final phase of Babylonian writing we do not find any examples of zero signs at the end of numbers. Though there are many instances of cases like .,20 there is no safe example of a writing like 20,. known to me. In other words, in all periods the context alone decides the absolute value of a sexagesimally written number. In Old-Babylonian mathematical texts we find several cases where a final result was written by means of individual symbols for the fractions, e.g. 1,30 might be called

'1 and ½' which shows that we should transcribe $1;30 = 1\frac{1}{2}$ and not $1,30 = 90$.

The ambiguity with respect to fractions and integers is of no importance for the practice of computation. Exactly as we multiply two numbers regardless of the position of the decimal point, one can also operate with the Babylonian numbers and determine the absolute value at the end if necessary. For the numerical process itself it is indeed a great advantage that one does not need to worry about special values for fractions and integers. It is precisely this feature which gave the Babylonian system its tremendous advantage over all other number systems in antiquity.

16. The mathematical texts can be classified into two major groups: 'table texts' and 'problem texts'. A typical representative of the first class is the multiplication table discussed above (p. 24). The second class comprises a great variety of texts which are all more or less directly concerned with the formulation or solution of algebraic or geometrical problems.

17. The table texts allow us to reconstruct a small, however insignificant, bit of historical information. The archives from the city of Nippur, now dispersed over at least three museums, Philadelphia, Jena and Istanbul, have given us a large percentage of table texts, many of which are clearly 'school texts', i.e. exercises written by apprentice scribes. This is evident, e.g. from the repetition in a different hand of the same multiplication table on obverse and reverse of the same tablet. Often we also find vocabularies written on one side of a tablet which shows mathematical tables on the other side. These vocabularies are the backbone of the scribal instruction, necessary for the mastery of the intricacies of cuneiform writing in Akkadian as well as in Sumerian. Finally, many of our mathematical tables are combined with tables of weights and measures which were needed in daily economic life. There can be little doubt that the tables for multiplication and division were developed simultaneously with the economic texts. Thus we find explicitly confirmed what could have been concluded indirectly from our general knowledge of early Mesopotamian civilization.

Ignoring variations in small details, tables of reciprocals are

lists of numbers as follows:

2	30	16	3,45	45	1,20
3	20	18	3,20	48	1,15
4	15	20	3	50	1,12
5	12	24	2,30	54	1,6,40
6	10	25	2,24	1	1
8	7,30	27	2,13,20	1,4	56,15
9	6,40	30	2	1,12	50
10	6	32	1,52,30	1,15	48
12	5	36	1,40	1,20	45
15	4	40	1,30	1,21	44,26,40

. . . Thus our stock of multiplication tables is not a collection of tables for all products $a.b$, for a and b from 1 to 59, but tables for the products $a.\bar{b}$ where $\bar{b}$ is a number from the right-hand side of our last list. The character of these numbers $\bar{b}$ is conspicuous enough; they are the reciprocals of the numbers b of the left column, written as sexagesimal fractions:

$$1/2=0;30$$
$$1/3=0;20$$
$$1/4=0;15$$
$$\text{etc.}$$
$$1/1,21=0;0,44,26,40.$$

We can express the same fact more simply and historically more correctly in the following form. The above 'table of reciprocals' is a list of numbers, b and $\bar{b}$, such that the products $b.\bar{b}$ are 1 or any other power of 60. It is indeed irrelevant whether we write

$$2.30=1,0$$
or $$2.0;30=1$$
or $$0;2.30=1$$
or $$0;2.0;30=0;1 \text{ etc.}$$

Experience with the mathematical problem texts demonstrates in innumerable examples that the Babylonian mathematicians made full use of this flexibility of their system.

Thus we have seen that the tables of multiplication combined with the tables of reciprocals form a complete system, designed to compute all products $a.\bar{b}$ or, as we now can write, all sexagesimal divisions a/b within the range of the above-given table

of reciprocals. This table is not only limited but it shows gaps. There is no reciprocal for 7, for 11, for 13 or 14, etc. The reason is obvious. If we divide 7 into 1 we obtain the recurrent sexagesimal fraction 8,34,17,8,34,17 . . .; similarly for 1/11 the group 5,27,16,21,49 appears in infinite repetition. We have tables which laconically remark '7 does not divide', '11 does not divide', etc. This holds true for all numbers which contain prime numbers not contained in 60, i.e. prime numbers different from 2, 3 and 5. We shall call these numbers 'irregular' numbers in contrast to the remaining 'regular' numbers whose reciprocals can be expressed by a sexagesimal fraction of a finite number of places. . . .

The limitations of the 'standard' table of reciprocals which we reproduced above (p. 27) did not mean that one could not transgress them at will. We have texts from the same period teaching how to proceed in cases not contained in the standard table. We also have tables of reciprocals for a complete sequence of consecutive numbers, regular and irregular alike. The reciprocals of the irregular numbers appear abbreviated to three or four places only. But the real expansion came in the Seleucid period with tables of reciprocals of regular numbers up to 7 places for b and resulting reciprocals up to 17 places for $\bar{b}$. A table of this extent, containing the regular numbers up to about 17.10^{12} can be readily used also for determining approximately the reciprocals of irregular numbers by interpolation. Indeed, in working with astronomical texts I have often used this table exactly for this purpose and I do not doubt that I was only repeating a process familiar to the Seleucid astronomers.

MATHEMATICAL CUNEIFORM TEXTS

edited by O. Neugebauer and A. Sachs

CHAPTER 3. PROBLEM-TEXTS

§1. *Introduction*

The Old-Babylonian texts vary greatly as to type and content. Some tablets contain only one example, which gives all the details of the solution of the problem stated at the beginning of the text. We even have tablets which contain only a part of the working out of the solution of a problem. At the other extreme are tablets which state hundreds of problems in a very condensed form but give no answers. Between these two extremes lie all sorts of intermediate types: texts with two or more examples which are worked out in detail and which are well arranged according to the degree of mathematical complication, texts with many examples of quite diverse character arranged very carelessly, and texts which represent a smaller collection of coordinated problems.

The text (§2) tabulates the answers to a problem containing Pythagorean numbers (or Pythagorean triangles). It is the oldest preserved document in ancient number theory.

§2. *Pythagorean Numbers. A. Plimpton 322*

Obverse	I	II	III	IV
1	[*ta-k*]*i-il-ti si-li-ip-tim*	ib-si$_8$ sag	ib-si$_8$ *si-li-ip-tim*	mu-bi-im
2	[*sa in-*]*na-as-sa-bu-u-*[*m*]*a sag i-*	. . . *-u*		
3	[1,59], 15	1,59	2,49	ki-1
4	[1,56,56],58,14,50,6,[1]15	56,7	3,12,1[2]	ki-2
5	[1,55,7],41,15,33,45	1,16,41	1,50,49	ki-3
6	[1],5[3,1],0,29,32,52,16	3,31,49	5,9,1	ki-4
7	[1],48,54,1,40	1,5	1,37	ki[-5]
8	[1],47,6,41,40	5,19	8,1	[ki-6]
9	[1],43,11,56,28,26,40	38,11	59,1	ki-7

1. 50,6 written like 56.
2. 3,12,1 error for 1,20,25.

Plate 3. Plimpton 322. Pythagorean Numbers.

10 [1],41,33,59,3,45	13,19	20,49	ki-8
11 [1],38,33,36,36	9,1[1]	12,49	ki-9
12 [1],35,10,2,28,27,24,26,40	1,22,41	2,16,1	ki-10
13 [1],33,45	45	1,15	ki-11
14 [1],29,21,54,2,15	27,59	48,49	ki-12
15 [1],27,3,45	7,12,1[2]	4,49	ki-13
16 [1],25,48,51,35,6,40	29,31	53,49	ki-14
17 [1],23,13,46,4[0]	56	53[3]	ki[-15]

COMMENTARY

a. Description of the Tablet. In its present state, the tablet represents the right-hand part of a larger text. The presence of modern glue, until the recent baking of the tablet, on the left (broken) edge shows that the missing part must have been lost after the excavation of the tablet. The size of the preserved part, $4\frac{7}{8}$ by $3\frac{3}{8}$ in. (12·7 by 8·8 cm), would make it unlikely that much more than half (or even less) of the existing part is missing. The reverse is uninscribed.

The script is clearly Old-Babylonian, i.e., it falls in the period between 1900 and 1600 B.C. The sign for 9 consists of three superimposed rows of three vertical wedges each. Zeros are not indicated by a special sign, but a blank space occurs in lines 3 and 15 where zero is called for; on the other hand, lines 7, 8, 10, etc., show that a blank space does not necessarily indicate zero.

b. Content. The text deals with 'Pythagorean triangles': right triangles whose sides are integers. Let l denote the longer, b the shorter leg of a right triangle, d its hypotenuse; then l, b and d are integers which fulfil the relation

$$l^2+b^2=d^2 \tag{1}$$

The values of d and b for 15 such Pythagorean triangles are given in the second and third preserved columns of our text. One might assume that the missing part contained the corresponding

1. 9,1 error for 8,1.
2. 7,12,1 (the square of 2,41) error for 2,41.
3. 53 error for 1,46 (i.e., 2.53).

values of l. The first preserved column gives the ratios of d^2 to l^2. Although the values of d and b vary in a very irregular manner, the ratios $\frac{d^2}{l^2}$ decrease almost linearly. Because the difference from line to line is very small (average 0;2,34), this is virtually equivalent to saying that $\frac{d}{l}$ decreases almost linearly (average difference 0;0,59,17, . . . i.e. almost 0;1), and we shall see that this is indeed the proper formulation of the problem. Formulating the problem with respect to the triangles we can say that we start out with almost half a square (because the value of b: l which corresponds to the first line [line 3] is 0;59,30) and gradually diminish the angle between l and d step by step, the lowest value being almost exactly 31°. It must, however, be kept in mind that the actual size of these triangles varies considerably owing to the fact that all sides are integral solutions of (1) and not approximations.

The numbers 1, 2, . . ., 15 in the last column have no exact relationship to the preceding numbers but merely indicate the number of the steps.

c. Method of Solution. We turn now to the question of great historical interest: How were the mathematicians of the Old-Babylonian period able not only to solve the Pythagorean equation (1) in integers but to adapt the solutions to the further condition that the proportion $\frac{d}{l}$ decrease from step to step by a number deviating very little from one-sixtieth?

In order to answer this question, we must first compare the solutions for l, b and d given in the tablet. These are listed in the following table with the correction of the four errors in lines 4, 11, 15 and 17 mentioned in the footnotes to the transcription:

Line	l	b	d
3	2, 0	1,59	2,49
4	57,36	56, 7	1,20,25
5	1,20, 0	1,16,41	1,50,49
6	3,45, 0	3,31,49	5, 9, 1
7	1,12	1, 5	1,37
8	6, 0	5,19	8, 1

Line	*l*	*b*	*d*
9	45, 0	38,11	59, 1
10	16, 0	13,19	20,49
11	10, 0	8,1	12,49
12	1,48, 0	1,22,41	2,16, 1
13	1, 0	45	1,15
14	40, 0	27,59	48,49
15	4, 0	2,41	4,49
16	45, 0	29,31	53,49
17	1,30	56	1,46

Before proceeding, it is necessary to note in the first place that line 13 gives the 15-fold values of the triangle with the sides $l=4$, $b=3$, $d=5$; and secondly, that the last line contains the common factor 2. All the other solutions are relatively prime.

The most important feature in this table is the obvious difference in character of the numbers l on the one hand and the b's and d's on the other: the b's and d's are 'complicated' numbers, but the l's are very 'simple'. This off-hand impression can be translated into precise terms by using the well known theorem[1] that all relatively prime Pythagorean numbers are contained exactly once in the set of numbers

$$(2) \qquad l=2pq \qquad b=p^2-q^2 \qquad d=p^2+q^2$$

where p and q are relatively prime integers, both not being simultaneously odd and $p>q$. If one calculates the numbers p and q which bring our numbers l, b and d to the form (2), one will find that the p's as well as the q's are 'regular numbers'. In other words, the p's and q's can be characterized by three numbers α, β and γ, the exponents of 2, 3, and 5, respectively. If we write

$$2^{\alpha}3^{\beta}5^{\gamma}=(\alpha, \beta, \gamma),$$

we then obtain the following list of numbers p and q which satisfy (2):[2]

1. Kronecker, Zahlentheorie p. 31.

2. We include here the values from lines 13 and 17 although a common factor 15 of l, b and d occurs in line 13, and p and q are both odd in line 17.

Line	*p*	*q*
3	12=(2,1,0)	5=(0,0,1)
4	1, 4=(6,0,0)	27=(0,3,0)
5	1,15=(0,1,2)	32=(5,0,0)
6	2, 5=(0,0,3)	54=(1,3,0)
7	9=(0,2,0)	4=(2,0,0)
8	20=(2,0,1)	9=(0,2,0)
9	54=(1,3,0)	25=(0,0,2)
10	32=(5,0,0)	15=(0,1,1)
11	25=(0,0,2)	12=(2,1,0)
12	1,21=(0,4.0)	40=(3,0,1)
13	2=(1,0,0)	1=(0,0,0)
14	48=(4,1,0)	25=(0,0,2)
15	15=(0,1,1)	8=(3,0,0)
16	50=(1,0,2)	27=(0,3,0)
17	9=(0,2,0)	5=(0,0,1)

The sense in which the numbers $l=2pq$ are 'simple' is now made clear by this list: they are numbers of the form (α, β, γ), i.e. so-called 'regular numbers' or numbers whose reciprocals are *finite* sexagesimal fractions.[1]

This latter quality of the p's and q's also yields the answer to the question how the numbers of our list were found which not only solve equation (1) but also furnish within narrow limits given proportions $\frac{d}{l}$. Using (2), we find for this proportion

$$\frac{d}{l}=\tfrac{1}{2}(p.\bar{q}+q.\bar{p}) \tag{3}$$

where $\bar{p}$ and $\bar{q}$ represent the reciprocals of p and q, respectively. In other words, our problem *requires* that p *and* q be regular numbers in order to obtain expressions (3) with finite sexagesimal fractions.

We can say more. With the single exception of $2,5=5^3$, all the numbers p and q of our list are contained in the group of regular numbers which constitute the 'reciprocal tables'. . . .

Our final result, then, is that our tablet was calculated by selecting numbers $p\bar{q}$ and $q\bar{p}$ from combined multiplication tables such that (3) has a value as near as possible to the required values

1. Cf. e.g. Neugebauer, *Vorlesungen*, pp. 6 and 12 ff.

of $\frac{d}{l}$; Pythagorean numbers were then formed with these values of p and q according to (2).

d. Historical Consequences. We now see that Babylonian number theory was acquainted with rules like (2) to produce Pythagorean numbers, i.e. a theorem like Euclid X 29 lemma 1.

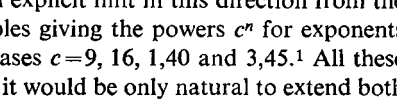
There can be little doubt that the Pythagorean numbers did not remain the only problem treated by this part of Babylonian mathematics. We have an explicit hint in this direction from the extant material itself: tables giving the powers c^n for exponents $n=1, 2, \ldots, 10$ for the bases $c=9$, 16, 1,40 and 3,45.[1] All these are regular numbers, and it would be only natural to extend both problems and methods as described above to combinations of other numbers and different exponents. The study of sequences Σn, Σn^2 etc.[2] points in the same direction. Details can be disclosed only by the discovery of new texts, but their general direction seems evident.

In summary, our text gives the final link which connects the different parts of Old-Babylonian mathematics by the investigation of the fundamental laws of the numbers themselves.

Ca MLC 1950.
(Transcription of Figure 1.1)

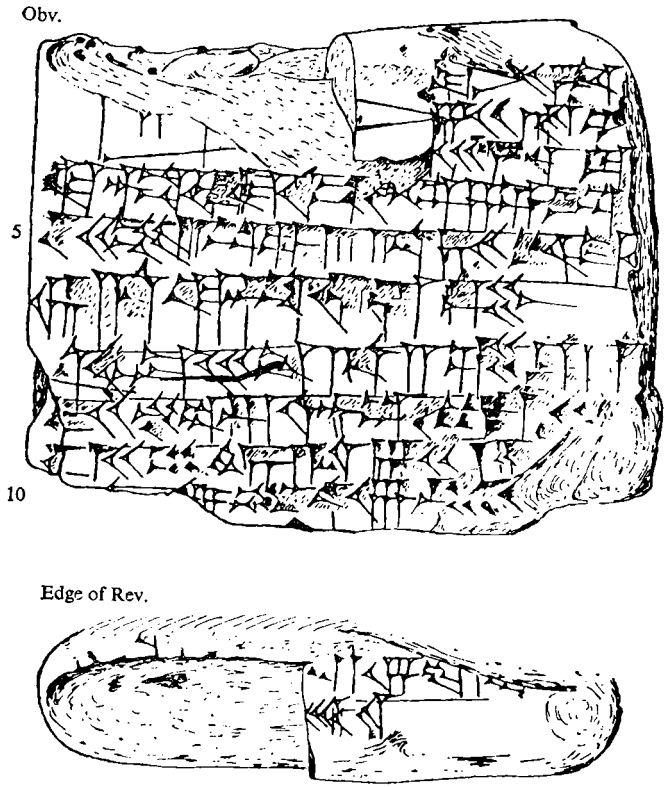
Obverse

1 sag-dù 20 GAR uš
2 5, 20 a-šà-bi
3 30 GAR . . . LU(?)
4 sag-an-ta *ù* sag-ki-ta *mi-nu-um* za-e kì-ta[-zu-dè]



5 igi 20 du_8 3 *ta-mar* 3 *a-na* 5,20 *i-ši-ma* 1[6]
6 16 *a-na* sag-an-ta ù . . .-. . .-LU-*bu*
7 . . .-. . .-*ma* 30 uš *a-na* 2 *e-ṣí*(?)-*ip*[3] 1

1. MKT I, pp. 77 ff.
2. MKT I, pp. 102 f., 497 f., and MKT III, pp. 13 f.
3. This reading, if correct, presupposes that the scribe inverted the order of the two components of the sign ZI.



Ca. MLC 1950

Fig. 1.1. Ca. MLC 1950.

8 *ù* 20 *mu-ta-ri-tam* an-ta UL-gar 1,20
9 igi 1,20 du$_8$ 45 *a-na* 5,20 a-šà [*i-ši-ma* 4]
10 [4 *a-na*] 16 daḫ *i-na* 16 zi 2[0 sag-an-ta]
11 [12 sag-ki-ta]
Reverse destroyed except for traces at end

TRANSLATION

Obverse

1 A triangle. 20 GAR is the length
2 5,20 its area,
3 30 GAR the . . .
4 What are the upper width and the lower width? [When you] perform (the operations),
5 take the reciprocal of 20, (and) you will see 0;3. Multiply 0;3 by 5,20, and (the result is) 1[6].
6 16 to (?) the upper width and . . .
7–8 . . . 30, the length, multiply by 2, add (the resulting) 1,0 and 20, the upper perpendicular, (and the result is) 1,20.
9 Take the reciprocal of 1,20, [multiply] (the resulting) 0;0,45 by 5,20, the area, [and (the result is) 4].
10 Add [4 to] 16, subtract from 16. [The upper width is] 2[0];
11 [the lower width is 12.]
Reverse destroyed except for traces at the end.

COMMENTARY

The problem presented here belongs to a class which is well attested: a triangle is subdivided by a line parallel to the base into a trapezoid and a smaller triangle. One group of dimensions

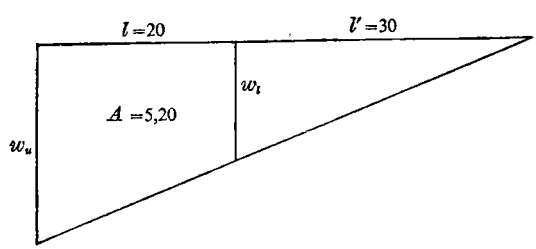


Fig. 1.2.

is given; find the others from the given relations. In our particular case, we are given a triangle, the altitude 50, subdivided into two parts $l=20$ and $l'=30$.[1] In addition, the area $A=5,20$ of the trapezoid is given. Find the lengths w_u and w_l of the parallel sides of this trapezoid.[2]

The method of solving the problem in question consists in finding both $\frac{1}{2}(w_u+w_l)$ and $\frac{1}{2}(w_u-w_l)$ such that the sum and difference of these expressions give the answer. The first expression is easily found. From the formula

$$A=\tfrac{1}{2}(w_u+w_l)l \tag{1}$$

or the area of the trapezoid, it follows that we have

$$\frac{A}{l}=\frac{5,20}{20}=5,20.0;3=16=\tfrac{1}{2}(w_u+w_l), \tag{2}$$

as given in the text.

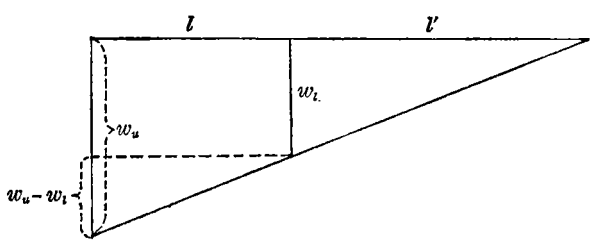


Fig. 1.3.

The next step requires explanation. We introduce the dotted line as given in Figure 1.3 and then find, from similar triangles,

$$\frac{w_u}{l+l'}=\frac{w_l}{l'}=\frac{w_u-w_l}{l}$$

or

$$w_u=\frac{l+l'}{l}(w_u-w_l) \qquad w_l=\frac{l'}{l}(w_u-w_l).$$

1. For the sake of convenience, a right triangle is assumed in our commentary. It is equally possible to assume an isosceles triangle, each side 50. One would then also have to assume the approximate formula for the area of a trapezoid.

2. The left side is, as usual, called the 'upper width' because we must turn the figure 90° in clockwise direction.

By adding, we obtain

$$w_u+w_l=\frac{2l'+l}{l}(w_u-w_l)$$

and consequently

$$\tfrac{1}{2}(w_u-w_l)=\frac{(w_u+w_l)l}{2(2l'+l)}.$$

Using (1), we can therefore write

(3) $$\tfrac{1}{2}(w_u-w_l)=\frac{A}{2l'+l}$$

and this is precisely the relation used by the scribe. He computes:

$$2l'=2.30=1,0 \qquad 2l'+l=1,0+20=1,20$$

$$\frac{1}{2l'+l}=0;0,45$$

and therefore

(4) $$\frac{A}{2l'+l}=0;0,45.5,20=4=\tfrac{1}{2}(w_u-w_l).$$

The relations (2) and (4) now provide the final answer

$$\tfrac{1}{2}(w_u+w_l)+\tfrac{1}{2}(w_u-w_l)=16+4=20=w_u$$

and

$$\tfrac{1}{2}(w_u+w_l)-\tfrac{1}{2}(w_u-w_l)=16-4=12=w_l.$$

Only parts of two lines of another problem are preserved on the reverse.

Oa. YBC 7284

YBC 7284 is a small, round tablet which reveals the existence of a coefficient accepted in the Old-Babylonian period for the transformation of the volume of bricks to units of weight.

On the reverse (convex side), we read:

1 sig_4 (One brick.)
ki-lá-bi en-nam (What is its weight?)
ki-lá-bi 8½ ma-na (Its weight is 8½ ma-na.)

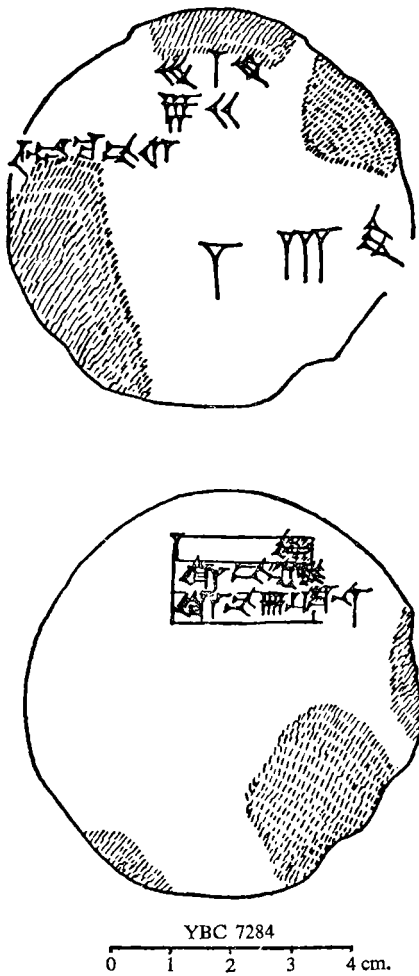


Fig. 1.4. YBC 7284.

YBC 4652

Problem-Text R

TRANSLATION

Obverse

.....

3 [1] [I found a stone], (but) [did n]ot we[igh it; .....]
[2] [I weighed (it): 1 ma-]na. [What was] the origin(al weight) of the sto[ne? .....]

4 [3] [I fou]nd [a stone], (but) did [not] we[igh it; .....]
[4] one-eleventh and [....., I weighed (it): 1 ma-na].
[5] [What] was the origin(al weight) of the stone? [.....]

5 [6] I found a stone, [(but) did not] we[igh it; .....],
[7] I weighed (it): 1 ma-na. [What was the] origin(al weight) of the stone? [.....]

6 [8] I found a stone, [(but) did not] we[igh it; (after) .....],
[9] I weighed (it): 1 ma-na. What [was the origin(al weight) of the stone]? The origin(al weight) of the sto[ne was .....]

7 [10] I found a stone, [(but) did not] weig[h it]; (after) I a[dd]ed one-seventh (and) a[dd]ed one-eleventh,
[11] I weighed (it): 1 ma-na. What was the origin(al) weight of the stone? The origin(al weight) of the stone was 2/3 ma-na, 8 gin (and) $22\frac{1}{2}$ se.

8 [12] I found a stone, (but) did not weigh it; (after) I subtracted one-seventh (and) subtracted one-thirteenth,
[13] I weighed (it): 1 ma-na. What was the origin(al weight) of the stone? The origin(al weight) of the stone was 1 ma-na (and) $15\frac{5}{6}$ gin.

9 [14] I found a stone, (but) did not weigh it; (after) I subtracted one-seventh, added one-eleventh, (and)
[15] subtracted one-thir[teenth], I weighed (it): 1 ma-na. What was the origin(al weight) of the stone?
[16] [The origin(al weight)] of the stone was 1 ma-na, $9\frac{1}{2}$ gin, (and) $2\frac{1}{2}$ se.

10 [17] [I fou]nd [a stone], (but) did not weigh (it); (after) I subtracted one-seventh, added one-eleventh,

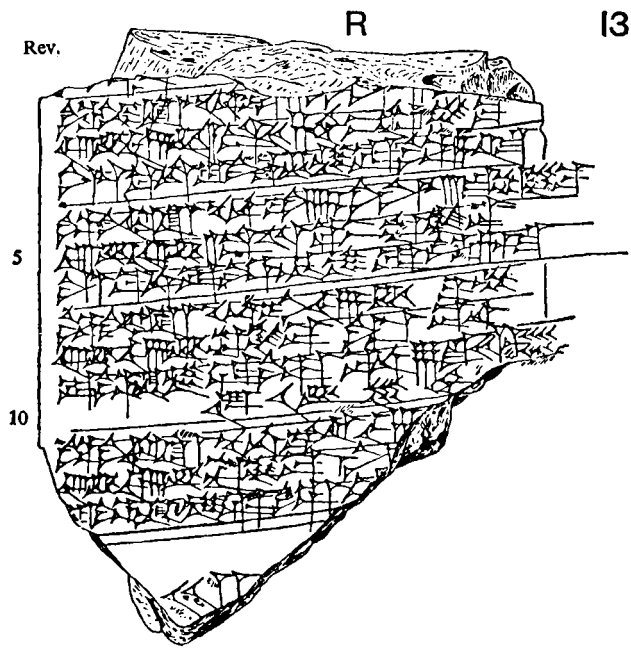


Fig. 1.5. YBC 4652 (reverse).

18 [.....] added [.....], I weighed (it): 1 ma-na. What was the origin(al weight) of the stone?

Reverse

.....

19 1 I found a stone, (but) did not weigh it; (after) I weighed (out) 6 times (its weight), [added] 2 gin, (and)
2 added one-third of one-seventh multiplied by 24,
3 I weighed (it): 1 ma-na. What was the origin(al weight) of the stone? The origin(al weight) of the stone was $4\frac{1}{3}$ gin.

20 4 I found a stone, (but) did not weigh it; (after) I weighed (out) 8 times (its weight), added 3 gin,
5 one-third of one-thirteenth I multiplied by 21, added (it), and then

Fig. 1.6. YBC 4652 (obverse).

6 I weighed (it): 1 ma-na. What was the origin(al weight) of the stone? The origin(al weight) of the stone was 4½ gin.

21 7 I found a stone, (but) did not weigh it; (after) I subtracted one-sixth (and)

8 added one-third of one-eighth, I weighed (it): 1 ma-na.

9 What was the origin(al weight) of the stone? The origin(al weight) of the stone was 1 ma-na, 9 gin, 21½ se,

10 and one-tenth of a se.

22 11 I found a stone, (but) did not weigh it; (after) I [.....] ⅔ of one-sixth [.....],

12 added one-third of one-eighth, I weighed (it): [1 ma-na].

13 What was the origin(al weight) of the stone? The origin(al weight) of the stone was 1 m[a-na . . .].

14 22 [sections].

COMMENTARY

a. Mathematical Commentary. According to the colophon, the tablet contained 22 examples when complete, but only 11 are even partly preserved; of these, 6 can be fully restored. All the problems of the tablet are obviously of the same type, resulting in a linear equation for one unknown quantity, the 'original' weight of a stone.

The preserved problems are the following:

No. 7

$$\left(x+\frac{x}{7}\right)+\frac{1}{11}\left(x+\frac{x}{7}\right)=1{,}0$$

Solution: $x=\frac{2}{3}$ ma-na $+8$ gin $+22\frac{1}{2}$ se $=48;7{,}30$ gin.

Indeed, $\frac{x}{7}=6;52{,}30 \qquad x+\frac{x}{7}=55 \qquad \frac{1}{11}\left(x+\frac{x}{7}\right)=5.$

No. 8

$$\left(x-\frac{x}{7}\right)-\frac{1}{13}\left(x-\frac{x}{7}\right)=1{,}0$$

Solution: $x=1$ ma-na $+15\frac{5}{6}$ gin $=1{,}15;50$ gin.

Indeed, $\frac{x}{7}=10;50 \qquad x-\frac{x}{7}=1{,}5 \qquad \frac{1}{13}\left(x-\frac{x}{7}\right)=5.$

No. 9

$$\left(x-\frac{x}{7}\right)+\frac{1}{11}\left(x-\frac{x}{7}\right)-\frac{1}{13}\left[\left(x-\frac{x}{7}\right)+\frac{1}{11}\left(x-\frac{x}{7}\right)\right]=1{,}0.$$

Solution: $x=1$ ma-na $+9\frac{1}{2}$ gin $+2\frac{1}{2}$ se $=1{,}9;30{,}50$ gin.

Indeed, $\frac{x}{7}=9;55;50 \qquad x-\frac{x}{7}=59;35$

$$\frac{1}{11}\left(x-\frac{x}{7}\right)=5;25$$

$$\frac{1}{13}\left[\left(x-\frac{x}{7}\right)+\frac{1}{11}\left(x-\frac{x}{7}\right)\right]=5.$$

No. 19

$$(6x+2)+\tfrac{1}{3}\cdot\tfrac{1}{7}\cdot 24(6x+2)=1,0$$

Solution: $x=4\tfrac{1}{3}$ gin.

Indeed, $6x+2=28$ $\quad \tfrac{24}{21}(6x+2)=32.$

No. 20

$$(8x+3)+\tfrac{1}{3}\cdot\tfrac{1}{13}\cdot 21(8x+3)=1,0$$

Solution: $x=4\tfrac{1}{2}$ gin.

Indeed, $8x+3=39$ $\quad \tfrac{21}{39}(8x+3)=21.$

No. 21

$$\left(x-\frac{x}{6}\right)+\tfrac{1}{3}\cdot\tfrac{1}{8}\left(x-\frac{x}{6}\right)=1,0$$

Solution: $x=1$ ma-na $+9$ gin $+21\tfrac{1}{2}$ se $+\tfrac{1}{10}$ se $+1,9;7,12$ gin.

Indeed, $\dfrac{x}{6}=11;31,12$ $\quad x-\dfrac{x}{6}=57;36$ $\quad \tfrac{1}{24}\left(x-\dfrac{x}{6}\right)=2;24.$

The remaining problems are completely broken away or too badly preserved to be restored with certainty.

The main difficulty encountered in interpreting the text of the problem consists in placing the parentheses correctly. The terminology alone is in itself inadequate; only experience with analogous problems,[1] when combined with the given solution, indicates the correct interpretation. The ancient scribes of course had the oral interpretation of their teachers at their disposal.

YBC 4612

Problem-Text S

TRANSLATION

Obverse

1 [1] 3,45 GAR is the length, 1,20 GAR the width. What is its area?

[2] Its area is 1 bur'u.

1. Cf. e.g. MKT I, p. 483.

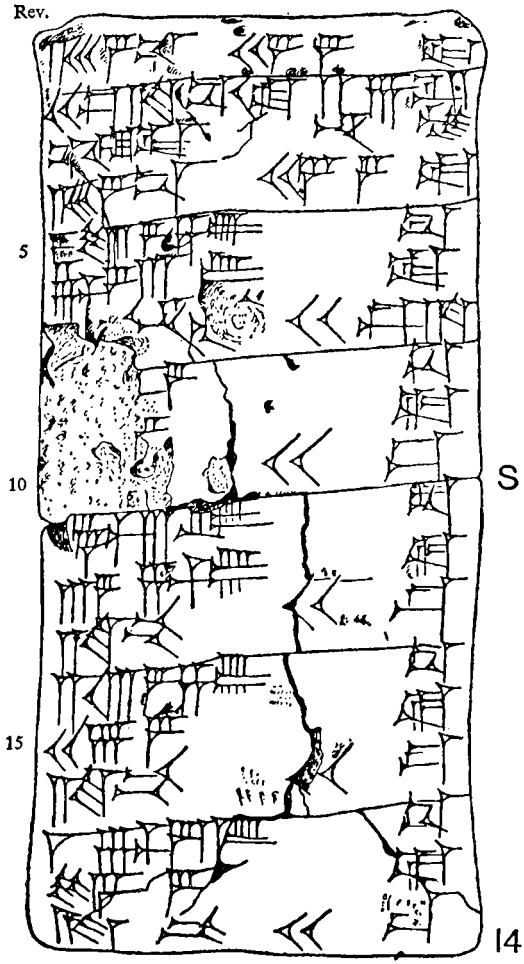


Fig. 1.7. YBC 4612 (reverse).

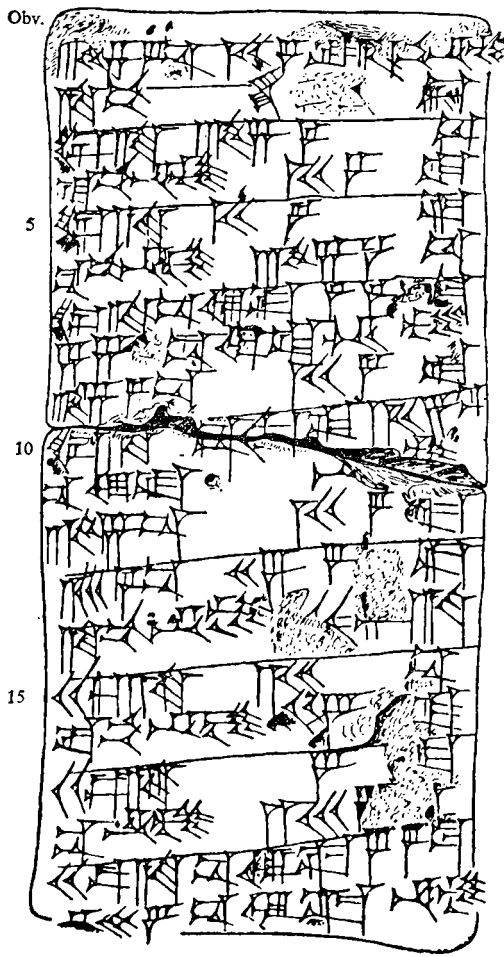


Fig. 1.8. YBC 4612 (obverse).

2 [3] 1 bur'u is the area, 3,45 GAR the length.
[4] What is its width? 1,20 GAR is the width.
3 [5] 1 bur'u is the area, 1,20 GAR the width.
[6] What is its length? 3,45 GAR is the length.
4 [7] 1 bur'u is the area; I added the length and the width, and (the result is)
[8] 5,5. What are the length and the width?
[9] 3,45 GAR is the length; 1,20 GAR is the width.
5 [10] 1 bur' u is the area; the length exceeded the width by 2,25.
[11] What are the length and the width?
[12] 3,45 (GAR) is the length; 1,20 GAR is the width.
6 [13] 2,30 GAR is the length, 24 GAR the width.
[14] What is its area? 2 bur is the area.
7 [15] 2 bur is the area, 2,30 (GAR) the length.
[16] What is its width? 24 (GAR) is the width.
8 [17] 2 bur is the area, 24 (GAR) the width.
[18] What is the length? 2,30 GAR is the length.
9 [19] 2 bur is the area; I added the length and the width, and (the result is)
[20] 2,54 (GAR). What are the length and the width?

Reverse

[1] 2,30 GAR is the length; 24 GAR is the width.
10 [2] 2 bur is the area; the length exceeded the width by 2,6 (GAR).
[3] What are the length and the width?
[4] 2,30 GAR is the length; 24 GAR is the width.
11 [5] 8,53 GAR (and) 4 kus is the length,
[6] 6½ GAR (and) 3 kus the width.
[7] What is its [area]? 2 bur is the area.
12 [8] [1,52]½ GAR is the length,
[9] [32] GAR the width. (What is its area?)
[10] Its [area] is 2 bur.
13 [11] 17½ <GAR>(and) 3⅓ kus is the length,
[12] 3 GAR (and) 4½ kus the width. (What is its area?)
[13] Its area is 2 bur.
14 [14] 2,13 GAR (and) 4 kus is the length.
[15] 27 GAR the width. (What is its area?)

16 Its area is 2 bur.

15 17 1,6½ GAR (and) 2 kus is the length,
18 54 GAR the width. (What is its area?)
19 Its area is 2 bur.

COMMENTARY

The mathematical aspect of this group of 15 problems is extremely simple. The only relation used is the fact that the area of a rectangle of sides x and y ($x>y$) is xy. The numerical values of the magnitudes used are

Nos. 1 to 5 x=3,45 GAR y=1,20 GAR
xy =5,0,0 GAR^2=1 bur'u

Nos. 6 to 10 x=2,30 GAR y=24 GAR
xy=1,0,0 GAR^2=2 bur

and the problems are:

Nos.	*given*	*find*
1 and 6	x, y	xy
2 and 7	xy, x	y
3 and 8	xy, y	x
4 and 9	$xy, x+y$	x, y
5 and 10	$xy, x-y$	x, y

The solution of the last two pairs requires quadratic equations of the standard type.

The last group (Nos. 11 to 15) is even simpler: find xy if x and y are given. The emphasis of these problems obviously lies in the metrological aspect. If we express the area in GAR^2, we find the value 1,0 GAR^2 for the area in No. 13, but 1,0,0 GAR^2 in the remaining four examples. The text, however, gives all five areas as 1,0,0 GAR^2.

Because the area is supposed to be 1,0,0 GAR^2 (i.e. 2 bur) in all five cases, x and y (expressed in GAR) are reciprocal numbers. The values are:

No.	x	y
11	8,53;20 GAR	6;45 GAR
12	[1,52;]30 GAR	[32] GAR
13	17;46,40 GAR	3;22,30 GAR
14	2,13;20 GAR	27 GAR
15	1,6;40 GAR	54 GAR

The restoration in No. 12 is uniquely determined by the fact that there is no other pair of numbers x and y such that $x.y=1,0,0$ and $x=a+0;30\ y=b$, where a and b are integers and where $x>y$.

As indicated above, the solution given in No. 13 is wrong because the product xy is only 1,0 not 1,0,0. This shows that the scribe's starting-point was a table of reciprocals but that he committed an error in determining the sexagesimal place. The correct value would have been 17,46;40 GAR = 17,46½ GAR 2 kus.

A 24194

—

Problem-Text T

TRANSLATION

42 [18]Area, 1 ese.
[19] One-fourteenth of the len[gth and the wid]th,
[20–21] and 2 times [that by which the length] exceeded the width; [I added 2,29];
[22] one-seventh of one-eleventh (thereof)
[23] I added to the length: 32.
43 [24] I multiplied by 2, I added: 34.
44 [25] I subtracted: 28.
45 [26] I multiplied by 2, I subtracted: [26].
46 [27] [I multiplied] by [1]5: [the length].
47 [28] I multiplied by 20: it exceeded by 10.
48 [29] I multiplied by 10: it was less by 10.
49 [30] I added to the width: 22.
119 [13] Area, 1 ese.
[14] One-half of [the length .....]
[15] [..... one-]sixth
[16] one[-.....] ..... 8 ..... I subtracted
[17] one[-.....] I added to the [len]gth: 32.
120 [18] [I multiplied by 2, I added: 3]4.
121 [19] I sub[tracted]: 28.
122 [20] [I multiplied by 2, I sub]tracted: 26.
123 [21] [I multipl]ied [by 15]: the length.
124 [22] I multiplied by 20: it exceeded by 10.
125 [23] I multiplied by 10: it was [less] by 10.

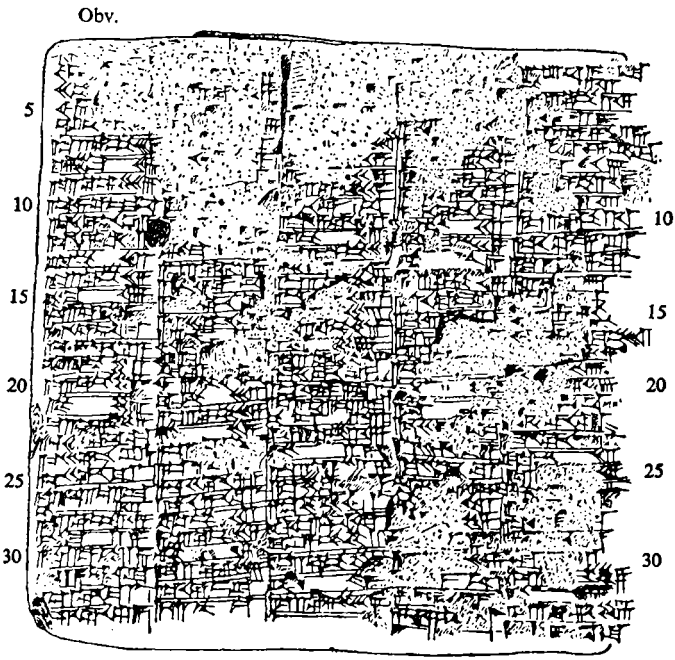


Fig. 1.9. A24194 (obverse).

126 [24] I added to the width: 22.
127 [25] [I multiplied] by 2, [I added]: 24.
139 [3] Area, 1 ese.
[4] I subtracted one-third from one-half the length,
[5] I added 25, one-seventh (thereof) and .....;
[6] I added one-eleventh thereof to the length: 32.
140 [7] I multiplied by 2, I added: 34.
141 [8] I subtracted: 28.
143 [10] I multiplied by 15: the length.
144 [11] I multiplied by 20: it exceeded by 10.
145 [12] I multiplied by 10: it was less by 10.
146 [13] I added to the width: 22.
147 [14] I multiplied by 2, I added: 24.

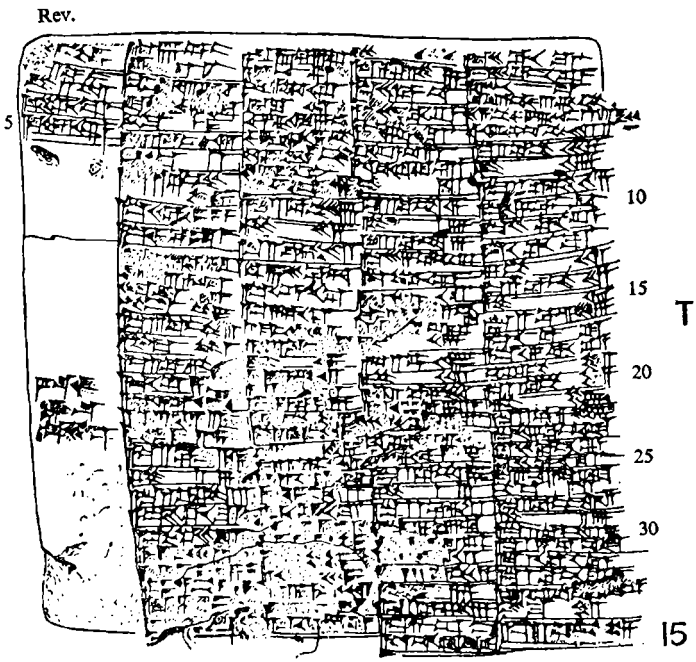


Fig. 1.10. A24194 (reverse).

148 [15] I subtracted: 18.
149 [16] I multiplied by 2, I subtracted: 16.
150 [17] I multiplied by 10: the width.
151 [18] I multiplied by 15: it exceeded by 10.
152 [19] I added to the length and the width: 52.

197 [[2]Ar[ea], 1 ese.
[3] The length and that by which the length
[4] exce]eded the width and
[5] one-fourth (thereof) and 6 I added together;
[6] I added 15 to one-eighth thereof;
[7] to one-eleventh thereof I added the length;
[8] one-eighth thereof
[9] I ad[ded to the length: 3]4.

198 [10] I multiplied by 2, I added: 38.
199 [11] I subtracted: 26.
200 [12] I multiplied by 2, I subtracted: 22.
201 [13] I multiplied by 7;30: the length.
202 [14] I multiplied by 10: it exceeded by 10.
203 [15] I multiplied by 5: it was less by 10.
204 [16] I added to the width: [24].
205 [17] I multiplied by 2, a[dded: 28].

COMMENTARY

a. General Commentary. This text, like the following one, can best be compared to an extensive collection of problems from a chapter of a textbook. It is obvious that a collection of this sort was used in teaching mathematical methods. They constitute a large reservoir of problems from which individual problems of any required type (say, speaking from a modern point of view, of a certain category of quadratic equations) could be selected.[1] This explains the schematic arrangement of the examples and made possible the employment of a terse style which, if isolated, would be extremely ambiguous.

The tablets are written with very small characters; this permitted the scribe to include 200 or more examples on a single tablet measuring about 3 by 4 inches. The difficulty in reading such small writing is considerably increased by the state of preservation, which is far from good. As a matter of fact, it is frequently only the systematic structure of the contents which makes it possible to interpret faint traces of signs.

We now turn to the explanation of the contents and the arrangement of the examples in our text, all of which deal with quadratic equations for two unknown quantities: 'length' (us, henceforth called x) and 'width' (sag, denoted by y). It is necessary to distinguish between two types of examples: 'main examples' and 'variants'. A characteristic case of a main example is the following: the first condition for the unknown quantities is that the 'area' (a-sa) is 1 ese = 10,0 GAR, i.e.

1. For a classification of Babylonian quadratic equations, see Gandz [1].

(1) $$xy=10,0.$$

Then a function $f(x,y)$ is defined by the expression

(2*a*) $$f(x,y)=1/11\{1/13[(x+4y)+2(x-y)]+45\}$$

and it is furthermore stated that

(2*b*) $$x+f(x,y)=35.$$

It is evident that (1) and (2) lead to a quadratic equation for x and y.

The 'variants' assume that (1) and (2a) are kept unchanged while (2b) is replaced by one of the following equations:

(2*c*)
$$\begin{aligned}
x+2f(x,y) &= 40\\
x-f(x,y) &= 25\\
x-2f(x,y) &= 20\\
x &= 6f(x,y)\\
x &= 8f(x,y)-10\\
y+f(x,y) &= 25\\
y+2f(x,y) &= 30\\
y-f(x,y) &= 15\\
&\text{etc.}\\
(x+y)+f(x,y) &= 55\\
(x+y)+2f(x,y) &= 1,0\\
&\text{etc.}\\
(x-y)+f(x,y) &= 15\\
&\text{etc.}
\end{aligned}$$

This list of variants shows the principle of arrangement: after a certain $f(x,y)$ is defined, it is added or subtracted with a factor 1 or 2 to or from x, y, $x+y$, $x-y$ etc.

A further observation can easily be made. If we compute x and y from (1), (2*a*) and (2*b*), or any of the equations (2*c*), we will always obtain

(3) $$x=30 \qquad y=20.$$

This statement can be immediately confirmed from our equations: it is evident that (1) is satisfied by (3); it follows from (2*a*) that $f(x,y)$ has the value 5 if we use (3); finally, it is again trivial that $x=30$ $y=20$ and $f(x,y)=5$ satisfy all equations (2*b*) or (2*c*).

The conditions (1) and (2) mentioned above are of course only

typical examples of a main problem and some of its variants. The fact remains, however, that the same set of values for x and y which is given by (3) satisfies all the problems of the two tablets discussed here as well as many tablets published previously. This clearly illustrates the procedure followed in composing these collections of problems. In order to guarantee real and positive solutions of the quadratic equations, one started from simple solutions like (3) and constructed the equations accordingly. All variants follow the same principle and their purpose lies exclusively in the teaching of methods to reduce problems of types (1) and (2) to the 'normal' form of quadratic equations which gives the product of the unknown quantities and their sum or difference. The factor ± 2 in the variants is of course chosen only to indicate any coefficient other than 1 in the previous variants, since a coefficient 1 can be overlooked in the numerical computation. The whole system of a main example with all its variants serves the purpose of giving the general rule of solution; this corresponds in a certain sense to an algebraic formula in which the letters can be replaced by special numbers in each individual case.

b. Mathematical Commentary. A 24194 contains 7 main examples and 240 variants. In 5 of the main examples (Nos. 1, 42, 81, 119, 139), the function $f(x,y)$ has the value 2 if we give x and y the values 30 and 20, respectively. The value 3 occurs in No. 54, the value 4 in No. 197. Unfortunately, the state of preservation of the text is such that only three of the main examples can be read or restored with some degree of probability. These are:

No. 42. $xy=10{,}0 \qquad x+f(x,y)=32,$

where $f(x,y)$ is an abbreviation for

$$1/7.1/11\{1/14(x+y)+2(x-y)+2{,}29\}=f(x,y).$$

The number 2,29 is, to be sure, restored; but this is necessary if the other readings are correct.

No. 139. The only restoration of this example which seems to conform to the preserved portion of the text is the following:

$$xy=10{,}0 \qquad x+f(x,y)=32,$$

where

$$f(x,y)=\frac{1}{11}\left\{\frac{1}{7}\left(\frac{x}{2}-\frac{1}{3}\frac{x}{2}+25\right)+17\right\}.$$

The number 17 which is added at the end is necessary to yield the correct result, but we do not see how this element of the formula was expressed in the text because we are unable to read the traces at the end of line 5.

No. 197. This is the only main example which is almost completely preserved in the text. We have

$$xy=10{,}0 \qquad x+f(x,y)=34,$$

where

$f(x,y)=1/8\{1/11[1/8[x+(x-y)+1/4(x+(x-y))+6]+15]+x\}$
with the value $f(30,20)=4$.

Problem-Text L

TRANSLATION

Obverse

1 [1] A little canal. 5 US is the length, 3 kus the width, 3 kus its depth, [10 gin] (volume) the assignment,

[2] 6 se (silver) the wages of a hired man. What are the area, the volume, the (number of) workers, the (total expenses in) silver?

[3] 1,15 SAR is the area; <2> iku (and) 25 SAR is the volume; 2 ges'u (and) 2,30 is the (number of) workers; [and 2/3] ma-na (and) 5 gin is the (total expenses in) silver.

2 [4] A little canal. 5 US is the length, 3 kus the width, 3 kus [its depth; (for the first) 1 kus] depth,[2] ⅓ ma-na (volume) is the *silutum*;

[5] (for the next) 2 kus depth[2] a volume of 10 gin is the dusu. [How much length did] one man take? 3 kus (and) 6 su-si.

3 [6] A little canal. 5 US is the length, 3 kus the width, 3 kus [its depth; (for the first) 1 kus] depth,[2] ⅓ ma-na[1] volume is the *silutum*;

1. For: SAR.

2. The text abbreviates the word in question by writing the first syllable only.

[7] (for the next) 2 kus depth,[2] a volume of 10 gin is the dusu. How much length did 30 workers take? They took 8 GAR length.

4 [8] A little canal. 5 US is the length, 3 kus the width, 3 kus its depth; (for the first) 1 kus depth,[2] ⅓ ma-na[1] volume is the *silutum*;

[9] (for the next) 2 kus depth,[2] a volume of 10 gin is the dusu. (In) how many days did 30 workers finish? They finished (after) 1 month (and) 7½ (days).[3]

5 [10] A little canal. 3 kus is the width, 3 kus its depth; (for the first) 1 kus depth,[2] <⅓ ma-na[1]> volume is the *silutum*; (for the next) 2 kus depth,[2] <a volume of 10 gin> is the dusu;

[11] 30 workers finished (after) 1 month (and) 7½ (days).[3] What is the length? 5 US is its length.

7 [15] A little canal. 5 US is the length, 3 kus the width, 3 kus its depth; (for the first) 1 kus depth,[2] ⅓ ma-na[1] (volume) is the *silutum*;

[16–18] (for the next) 2 kus depth,[2] (a volume of) 10 gin is the dusu. What fraction of a day did one man dig the *silutum*? What fraction of a day did he dig the dusu? And what is the volume? One-fifth of a day he extracted the *silutum*, (amounting to) 4 gin volume; 2/3 of a day and one-fifth of 2/3 of a day he dug the dusu, (amounting to) 8 gin volume.

8 [19] An old, little canal. 5 US is the length, 1 kus the width, 1 kus its depth;

[20] its increase cut off ½ kus of its *tarahhum*. What is its volume? ⅓ SAR (and) 5 gin.[4]

9 [21] An old, little canal. 5 US is the length, 1 kus the width, 1 kus its depth; its increase

[22] cut off ½ kus of its *tarahhum*;[2] ⅓ gin[1] (volume) is the assignment. How much length [did] one man [take? 4 GAR] length.

3. The text uses the ordinal for the length of time; to avoid awkwardness, this has been ignored in the translation.

4. Mistake for 25 S A R.

10 [23] An old, little canal. 5 US is the length, 1 kus the width, 1 kus its depth; [its increase]
[24] cut off $\frac{1}{2}$ kus of its *tarahhum*;[4] $\frac{1}{3}$ gin[3] (volume) is the assignment. [How much] length [did] 10 workers [take]?
[25] They took 40 GAR length.
11 [26] An [old], little [canal]. 5 US is the length, 1 kus the width, 1 kus [its depth];
[27] its increase cut off $\frac{1}{2}$ kus of its *tarahhum*;[4] [$\frac{1}{3}$ gin (volume) is the assignment].
[28] (In) how many days did 10 workers finish? [They finished] (after) $7\frac{1}{2}$ days.[3]

Reverse

12 [1] An old, little canal. 1 kus is the width, 1 kus its depth; its increase <cut off> $\frac{1}{2}$ kus of its *tarahhum*;[4] <$\frac{1}{3}$ gin is the assignment>.
[2] 10 workers finished (in) $7\frac{1}{2}$ days. What is its length? 5 US is [its length].
13 [3] An old, little canal. 5 US is its length, 2 kus the width, 1 kus its depth;
[4] its increase cut off $\frac{1}{2}$ kus of its *tarahhum*[4] (and)
[5] added 1 kus to its depth. What is its volume? 1 iku is the volume.
14 [6] An old, little canal. 5 US is its <length>, 2 kus the width, 1 kus its depth;
[7] its increase cut off $\frac{1}{2}$ kus of its *tarahhum*[4] (and) [add]ed 1 kus to its depth;
[8] $\frac{1}{3}$ gin[1] volume is the assignment. How much length did one man take?
[9] He took 1 GAR length.
15 [10] An old, little canal. 5 US is its length, 3 kus the width, 1 kus its depth;
[11–12] its increase cut off $\frac{1}{2}$ kus of its *tarahhum*[4] (and) added $1\frac{1}{2}$ kus to its depth. What is its volume?
16 [13] An old, little canal. 5 US is its <length>, 2 kus the width, 1 kus its depth;
[14] its increase cut off $\frac{1}{2}$ kus of its *tarahhum*[4] (and) added $1\frac{1}{2}$ kus to its depth;

15 (for the first) 1 kus of its depth, $\frac{1}{3}$ gin^3 volume is the assignment; (for the next) $1\frac{1}{2}$ kus of its depth, 10 gin volume is the .....[5]

16 How much length did one man take? He took $4\frac{1}{3}$ kus (and) 4 su-si length.

17 17-18 A little canal. 1 kus is the width, 1 kus its depth. What ..... waters of the watercourses .....

18 19-20 A little canal. 2 kus is the width, $1\frac{1}{2}$ kus its depth. What ..... waters of the watercourses .....

a. Mathematical Commentary. All the examples of this text deal with a little canal (pa_5-sig) of rectangular cross-section. The first seven assume a width of 3 kus and a depth of 3 kus or more. Nos. 8–16 have to do with an old canal (pa_5-sig-libirra) of similar dimensions which must be enlarged because of deposited silt or because a flow of water greater than that permitted by the original canal is required. The work involved therefore consists in cleaning out and/or expanding the existing canal. The last two examples apparently deal with the water supplied by the canal.

No. 1

The dimensions given for the length (us), width (dagal), and depth (bur) are

$$l = 5{,}0\ \text{GAR} \qquad b = 3\ \text{kus} = 0;15\ \text{GAR} \qquad h = 3\ \text{kus}$$

The work λ assigned to one man per day amounts to

$$\lambda = 0;10\ \text{SAR}$$

volume. The wages are

$$w = 6\ \text{se} = 0;2\ \text{gin silver}$$

per man per day.

The text requires the answers for: the area (gagar) $g = lb$ covered by the canal; its volume (sahar) V; the number m of workers (erim-ha) necessary to complete the whole work in one day, i.e., the total number of man-days; and, finally, the total expenses E for the work expressed in silver.

1. Unidentified sign, perhaps intended for dusu; one would expect the word for 'assignment'.

The answers given satisfy the following relations:

$$g = lb = 1{,}15 \text{ SAR}$$
$$V = lbh = 3{,}45 \text{ SAR} = [2]^1 \text{ iku } 25 \text{ SAR}$$
$$m = \frac{V}{\lambda} = 22{,}30 = 2 \text{ ges'u } 2{,}30$$
$$E = mw = 45 \text{ gin} = [2/3]^2 \text{ ma-na } 5 \text{ gin.}$$

The ges'u in the third line literally means ten-sixties, i.e., 600.

Nos. 2–5

The dimensions given for the canal in this group of four examples are the same as in No. 1, namely,

$$l = 5{,}0 \text{ GAR} \qquad b = 3 \text{ kus} = 0;15 \text{ GAR} \qquad h = 3 \text{ kus.}$$

The volume assigned to one man each day, however, is no longer 0;10 SAR but is raised to

$$\lambda = 0;12 \text{ SAR.}$$

This statement is not made explicitly, but the discussion of No. 7 will show that it is implicitly contained in the wording

1 kus *su* ⅓ ma-na (sahar) *si-lu-tum*
2 kus *su* sahar 10 gin dusu.

Anticipating this conclusion, we obtain the following results:

No. 2. The length l_λ which corresponds to one man's work per day is therefore given by

$$l_\lambda = \frac{\lambda}{bh} = \frac{0;12}{0;15.3} = \frac{4}{15} = 0;16 \text{ GAR} = 3 \text{ kus } 6 \text{ su-si.}$$

No. 3. The length of the canal worked by 30 men during one day is therefore

$$30.l_\lambda = 0;16.30 = 8 \text{ GAR.}$$

No. 4. The time required for 30 men to complete the total work on $l = 5{,}0$ GAR length is therefore

$$t = \frac{5{,}0}{8} = 37\tfrac{1}{2} \text{ days.}$$

This number is expressed as '1 month (and) $7\frac{1}{2}$ (days)'. The

1. Omitted by the scribe.
2. Restored.

assumption of 30 days as the length of one month for purposes of computation is well attested by other texts.[1]

No. 5. This problem is the inverse of No. 4: 30 men finish a total length of $l=5{,}0$ GAR in $37\frac{1}{2}$ days.

Nos. 8–12

Here begins a new type of problem characterized by the term pa_5-sig-libir-ra, 'an old, little canal'. The work no longer consists of digging out the entire profile of the canal but simply cleaning out the silted parts or perhaps even enlarging the existing profile beyond that of the canal when it was first dug. All five examples

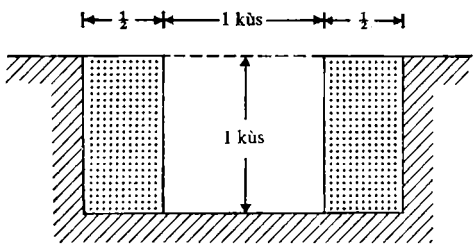


Fig. 1.11.

of this group assume dimensions for the existing profile as indicated in Figure 1.11, where the dotted area represents the part which is to be removed (called *tarahhu*):

No. 8 gives the dimensions of the free profile of the canal

$$l=5{,}0 \text{ GAR} \qquad b=1 \text{ kus}=0;5 \text{ GAR} \qquad h=1 \text{ kus}$$

and asks for the volume V^* to be dug if the width should be increased by

$$c=\tfrac{1}{2} \text{ kus}=0;2{,}30 \text{ GAR}$$

on each side. The answer should be

$$V^*=l.2c.h=5{,}0.0;5.1=25 \text{ SAR},$$

but by misinterpreting the order of magnitude, the scribe gives $\frac{1}{3}$ SAR 5 gin, i.e. 0;25 SAR.

No. 9 introduces the assignment of work per man per day.

$$\lambda=0;20 \text{ SAR},$$

1. Cf. e.g. MKT III, p. 63.

written, however, $\frac{1}{3}$ gin here and in the following examples where this value occurs. The values given for l, b, h and c are the same as above. The problem consists in finding the length dug by one man given by

$$\frac{\lambda}{2ch}=\frac{0;20}{0;5}=4 \text{ GAR.}$$

Nos. 10 and 11 make the same assumptions but ask for the length l_{10} for 10 men (hence 40 GAR is the result) and the time needed to complete the whole work by 10 men, which is

$$\frac{l}{l_{10}}=\frac{5,0}{40}=7\tfrac{1}{2} \text{ days.}$$

No. 12 is the inverse problem to No. 11, computing l from the number of men and the time.

Nos. 13–16

This new group not only assumes an increase of the 'old canal' in width but in depth as well (cf. Figure 1.12).

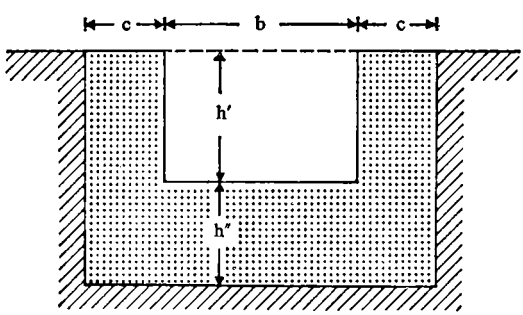


Fig. 1.12.

2

THE MOSCOW PAPYRUS

(c. 1850 B.C.)

THE Moscow Mathematical Papyrus, an ancient mathematical document, was discovered early in the 1890s in the Necropolis of Dra Abu'l Negga in Egypt. Shortly after the find, it was purchased by W. Golenishchev, a professor of Egyptian philology in the Egyptian University at Cairo. In 1912, after Professor Golenishchev's death, his entire collection including this papyrus, was acquired by the Moscow Museum of Fine Arts. Here it became the subject of intensive study by the curator of the Egyptological division and his associates. The painstaking efforts of these scholars and of other dedicated scientists drawn into the work by the world-wide interest in the papyrus, met with enormous difficulties. Numerous scribal errors and the loss of some fragments of the papyrus complicated the impediments normally inherent in the decipherment of an ancient language. After many years of work, the text was translated and accompanied by a detailed discussion of its contents.

The papyrus, which belongs to the latter part of the Middle Kingdom (2700–1670), is long and narrow, about eight centimetres in height. The text is in hieratic writing, underneath which the remnants of an earlier writing may be discerned. The Moscow Papyrus may be representative of a development in the method of writing, since the present writing is horizontal, reading from right to left, while the earlier partly erased writing is horizontal in some places and vertical in others. The textual material of the papyrus consists of twenty-five mathematical problems and their solutions, written in thirty-eight columns of varying length. The columns are well separated except in one instance where they run together. At the end of the last problem, a second long piece of papyrus is attached. Apart from some signs of an old writing remaining after an obviously incomplete erasure of a previous work, the attached papyrus is blank, offering mute evidence of the scribe's unfulfilled intention of writing a much larger work.

The problem types and methods of solution contained in the Moscow Papyrus are substantially the same as those of the Rhind and other

Egyptian mathematical papyri. However, the Moscow Papyrus exhibits some advance in the geometrical problems of which it contains four. As in the Rhind, the solutions of the geometrical problems are basically arithmetical, offering no trace of a proof in the Euclidean sense. An analysis of the mathematical background needed for the formulation of the solutions has shown that the problems presented in the Moscow Papyrus required greater knowledge and ingenuity than those of any other Egyptian mathematical works that have come down to us. In particular, problem 14 of the Moscow Papyrus, where the volume of the frustum of a square pyramid is found, represents the high point of ancient Egyptian mathematics.

FOUR GEOMETRICAL PROBLEMS FROM THE MOSCOW MATHEMATICAL PAPYRUS

by Battiscombe Gunn and T. Eric Peet

PROBLEM NO. 6

(*Transcription, Figure 2.1.*)

Example[1] *of calculating*[2] *an enclosure.*

If you are told: An enclosure of a set *and* 2 arurae, *the breadth having* $\frac{3}{4}$ *of the length:*

You are to treat $\frac{3}{4}$ *so as to find* 1; *the result is* $1\frac{1}{3}$. [*You are to take*] *this* [12] *which is in a* set *and* 2 arurae $1\frac{1}{3}$ <*times*>; *result* 16. *You are to calculate its square root; result;* 4 – *for the length, and the breadth has* $\frac{3}{4}$ *of it, namely* 3.

The correct procedure:

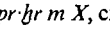
	4			
12	3	\1	4	
		\2	8	

1. To avoid confusion the numbers of the lines, given in the plates of transcriptions and referred to in the Notes on the Text, are omitted in the translations.

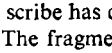
2. Literally, 'making' (*irt*). This verb is used of a number of various operations in the mathematical texts; in these four problems we find it meaning to 'work out' the volume or area of a figure; to 'take' a number so many times (multiplication) or 'take' a fraction of a number; to 'treat' a number so as to find another (division); to 'extract' a square root.

Notes on the Text

Line 1. The word for the subject of the calculation is damaged both here and in line 2. The sign preceding (the latter clear in both cases) will be , the word being shown by the following in line 2 to be feminine. suits the damaged signs perfectly in both places – judging by the photograph – and seems to be the only word that does so. The word is used of many kinds of rectangular enclosed spaces, cf. not only the meanings 'room', 'house', but also *ꜥt nt ḫt*, 'enclosure of trees', i.e. 'orchard'.[1]

Lines 2, 4.  . On the reading and meaning see the commentary.

Line 3. Before is a small gap, but it is unlikely that any signs are lost. For *ḫpr·ḫr m X*, cf. *Berlin Pap.* 6619 and the references *Journal*, XII, 125.

Line 4. The scribe has certainly omitted  before .

Lines 4, 5. The fragment now mounted at the beginnings of these lines is undoubtedly out of place. The restorations in the transcription, Pl. XXXV, suit the gaps and the sense.

Line 6. The hieratic sign for '8' at the end is quite abnormal (as are a number of this scribe's numerals), but the reading is certain.

Commentary

The purely mathematical content of the problem is simple and obvious. We are given a rectangular enclosure whose area is 12 square units and are told that its breadth is $\frac{3}{4}$ (the Egyptian has here, as always, $\frac{1}{2}$ $\frac{1}{4}$) of its length; find both.

The modern method of dealing with this problem is as follows:

Let x be the length in linear units; then $\frac{3}{4}x$ will be the breadth, and the area will be $\frac{3}{4}x^2$ square units. Equating this with the given area of 12 we get $\frac{3}{4}x^2 = 12$, or $x = 4$.

If the problem were set as an arithmetical exercise we should solve it in just the same way, while employing a more complicated phraseology to avoid the explicit introduction of the symbol x.

A glance at the Egyptian method will show that the arithmetical operations there performed are the same as our own. The $\frac{3}{4}$

1. Cf. Brugsch, *Worterb.*, Suppl., 185–6.

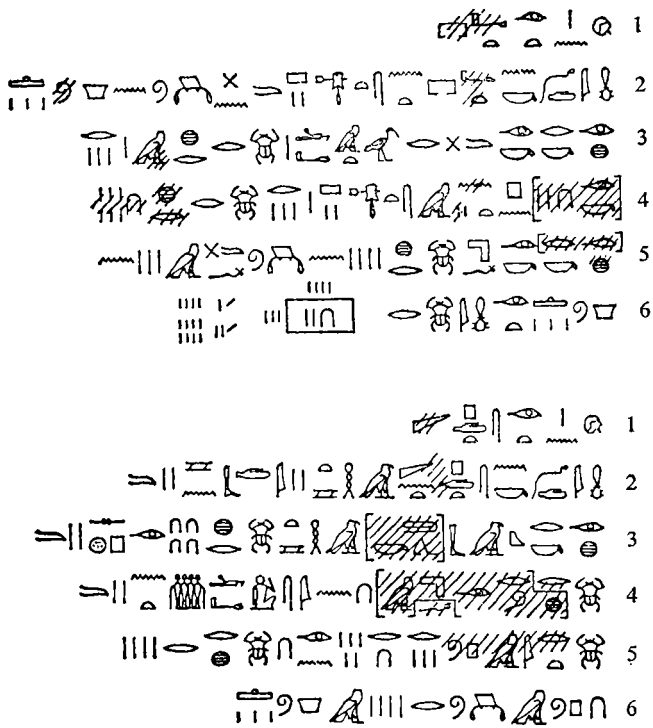


Fig. 2.1. Moscow Mathematical Papyrus. Problems 6, 7.

(written $\frac{1}{2}\ \frac{1}{4}$) is divided into unity[1] (i.e. inverted), giving $1\frac{1}{3}$, and 12 is multiplied by this, giving 16. The square root of the latter is then taken and stated, correctly, to be the length required. Finally

1. The Egyptian expression is here 'treat $\frac{1}{2}\ \frac{1}{4}$ so as to find 1'. To treat x so as to find y means that x is to be treated by multiplying by whole numbers or fractions, or both, until one or more of the products render y. See Peet, *Rhind Mathematical Papyrus*, 13–14. In this case the working would be:

$$\begin{array}{ll} 1 & \frac{1}{2}\ \frac{1}{4} \\ \frac{2}{3} & \frac{1}{3}\ \frac{1}{6} \text{ (on this stage see Peet, op. cit., 20.)} \\ \frac{1}{3} & \frac{1}{6}\ \frac{1}{12} \end{array}$$

Since $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$ and $\frac{1}{12}$ together render 1, the answer is $1\frac{1}{3}$.

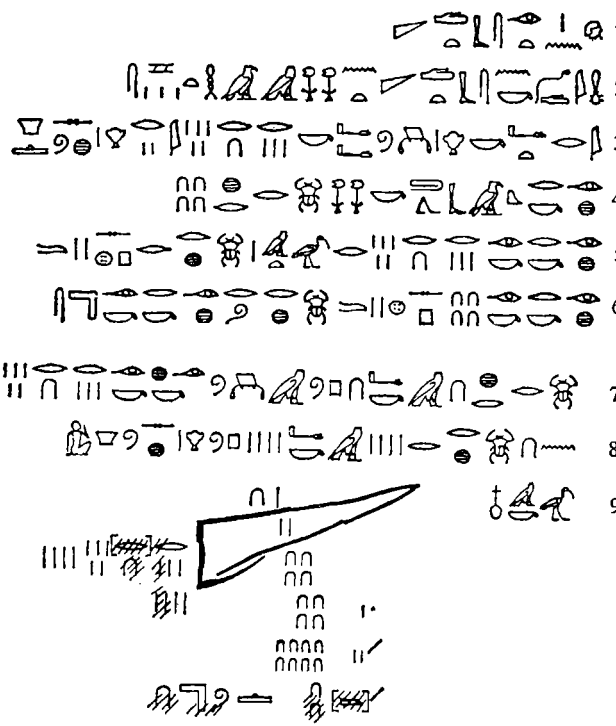


Fig. 2.2. Moscow Mathematical Papyrus. Problem 17.

the breadth, 3, is obtained as $\frac{3}{4}$ of this. This close correspondence with the modern method need cause no surprise, for after all there is but one way of obtaining the correct answer from the data given.

At the same time it is obvious that the mental process involved in the Egyptian method is not the same as that involved in ours. This is hardly the place in which to discuss the psychological meaning of the use of the unknown x in mathematics, but it is at least clear that this, whether used explicitly in an algebraical solution or implicitly in some purely 'arithmetical' one, gives

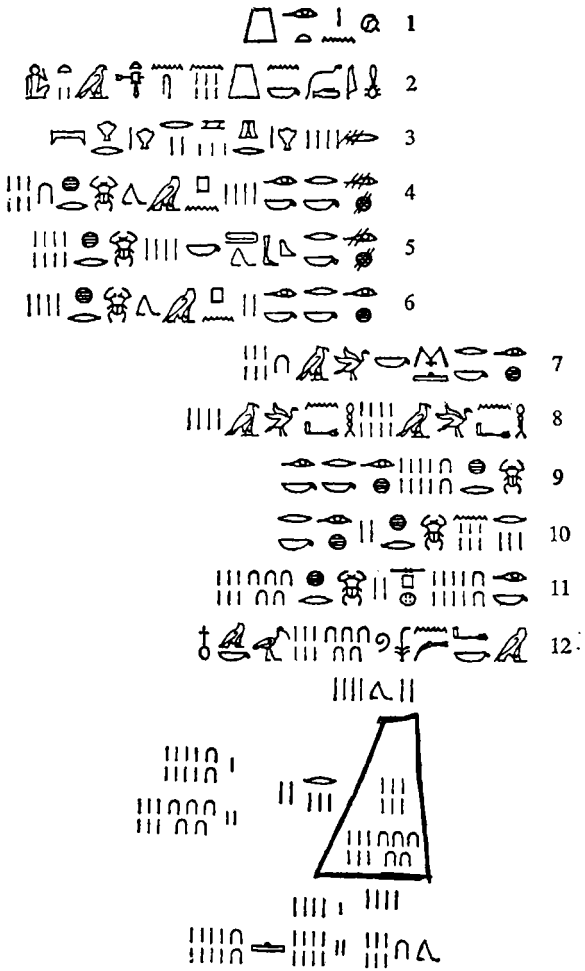


Fig. 2.3. Moscow Mathematical Papyrus. Problem 14.

to the modern method an abstract character entirely foreign to Egyptian mathematics. Once we have named our two sides x and $\frac{3}{4}x$ and multiplied them together to get the area – a concrete enough process this – we are in the realm of pure and abstract mathematics until we have found the value of x, namely 4, when we look back to the beginning to see what x was and so formulate our answer.

The Egyptian method, if we understand it rightly, is much more concrete than this. As usual, the solution shown gives only the operations to be performed. In the following paragraph we attempt to reconstruct the reasoned Egyptian solution of this problem, those parts which are found in the text being printed in italics:

'A rectangle having breadth $\frac{3}{4}$ of the length will have $\frac{3}{4}$ the area of a square on the length, since, one side of any rectangle being constant, the area varies directly as the length of the adjacent sides. The length of our enclosure will therefore be equal to the side of a square whose area shall have the same proportion to the area of the enclosure that the length of the latter has to its breadth. The breadth being $\frac{3}{4}$ of the length, *you must divide* $\frac{3}{4}$ *into* 1 to find the proportion of length to breadth; *the result is* $1\frac{1}{3}$. *The area of our enclosure, namely* 12, *is* therefore *to be multiplied by* $1\frac{1}{3}$; *result* 16, which is the area of the square sought. To find the length of a side of this square, *you are to calculate the square root*[1] *of* 16; *result* 4; *this is the length of the enclosure.*[2] *The breadth of the enclosure is* $\frac{3}{4}$ *of it* (the length), *namely* 3.'

1. The word for 'square root' is written here as elsewhere, with the sign , which represents either a 'corner' or more probably a 'right-angle'. The underlying idea is perhaps that a right-angle with equal arms, say of 3 in length, $\overset{3}{\urcorner}_3$, is the root of, in the sense of giving the *data* for, a square of 9. It is written in this MS, in *Berlin Pap.* 6619 and in *Kahun Papyri*, 8.40. If it is a masculine word it is probably to be read *tm*; cf. Schäfer in *Zeitschr. f. äg. Spr.*, XL, 96, title *ḫrf-(nf)-tm*.

2. The reckoner might equally well have elected to find the breadth first, it being the side of a square whose area is $\frac{3}{4}$ (Egyptian $\frac{1}{2}\ \frac{1}{4}$) of the given enclosure, i.e. 9 square units. The resulting 3 would actually have been found more simply than the length, for the division of unity by $\frac{3}{4}$ would have been avoided. This would, however, have had to be performed in the end in order to obtain the length 4 from the breadth 3.

The figure, with indications of length, breadth and area, and the multiplication of 4 by 3 which accompanies it,[1] constitute the proof, introduced here, as occasionally elsewhere, by *irt mi hpr*.[2]

So much for the mathematical aspect of the problem. We have now to examine the square and linear measures employed.

The area of the enclosure is given in lines 2 and 4 as . No word written with is known to us elsewhere as a measure.[3] The hieratic sign rendered by us is a short horizontal stroke, perhaps meant to be slightly curved in line 2, and straight (but inclining upwards to the right as do all 'horizontal' strokes in this script) in line 4;[4] this sign is identical with that used, with numerals under it, as here, in Middle Kingdom papyri and in the Rhind Papyrus for the *setat* (*arura*), a unit of area containing a square *khet*, the *khet* being a linear measure of 100 cubits; this sign is represented by , and the like in hieroglyphic. Now the group is clearly an equivalent of the '12' which is to be restored in line 4, for we there read '[you are to take] this [12] which is in , $1\frac{1}{3}$ <times>; result 16'. Whatever doubt may exist otherwise as to the exactitude of our restoration here, there can be none as to the '12', since the missing number, multiplied by $1\frac{1}{3}$, gives 16. The simplest interpretation of this difficult group would therefore seem to be that the stands, as elsewhere, for '2 *arurae*', and is preceded by a word , otherwise unknown, representing a unit of square measure equal to 10 *arurae*. The principal objection to this view is that it is impossible to understand why the usual term for a unit equal to 10 *arurae*, namely the 'thousand-of-land', is not used here, as it is elsewhere in this very papyrus.[5] That in both cases the word has no determinative and is not followed by the numeral '1'[6] could be put down to the ignorance of the scribe; the whole book abounds in errors both of orthography and in the forms of signs.

1. Once (4 is) 4; twice (4 is) 8; (once and twice = 3 times, 4 and 8 = 12).

2. See Peet, *Rhind Math. Pap.*, 23, and Gunn in *Journal*, XII, 126.

3. A word occurs in Gardiner, *Admonitions*, 14.4 (p. 90), and is tentatively translated by him 'ground'. It is certainly not used there as a measure.

4. The apparent turn downwards at the right-hand end in line 4 is probably a deception of the photograph, though it is impossible to be quite certain.

5. '2 thousands-of-land' is written in Problem No. 7 and in No. 17 (see below).

6. The omission of the numeral seems to be without parallel. The Egyptian for '1 cubit 3 palms', for instance, is , never .

PROBLEM NO. 7
(Transcription, Figure 2.1.)

Example of calculating a triangle.

If you are told: A triangle of 2 *thousands-of-land, the 'bank of* $2\frac{1}{2}$*':*

You are to double the area; result 40 (arurae). *Take (it)* $2\frac{1}{2}$ *times; result* [100. *Take its square root, namely*] 10. *Evoke* 1 *from* $2\frac{1}{2}$; *what results is* 2/5.[1] *Apply this to* 10; *result* 4. *It is* 10 (khet) *in length by* 4 (khet) *in breadth.*

Notes on the Text

Lines 1, 2. The determinative of *śpdt*, 'triangle', here, as in Problem No. 17 and the damaged problem No. 4, has a shape quite different from that of the sign with which the same word is written in the Rhind Papyrus. In the latter the sign is the symmetrical upright *point* (thorn?),[2] with apex at top, with which all forms and derivatives of *śpd* (primarily meaning 'to be sharp') are normally written in hieroglyphic and hieratic. In the Moscow Papyrus, on the other hand, it is a different sign, a scalene *triangle*, with vertical 'base' and the apex high up on the right.[3] Thus the word *śpdt*, 'the pointed', in its special meaning of 'triangle' here receives a new determinative, a triangle.

Line 3. The restoration of the determinatives of *kȝb* is based on the writings in Cols. 33 (Problem 17, line 4) and 27 (Problem 14, line 5).

Line 4. The restoration would just fill the gap. The tail of the is visible. The trace after suits (as in No. 17, line 6) and not . For the imperative *ỉr* in place of the more normal *ỉr·ḫr·k ỉr·k*, for which there is here no room, cf. lines, 3, 5 of this problem. Between and there is not room for more than , though one might expect as in the last problem.

Line 5. For *ḫprt ỉm pw* cf. Peet, op. cit., 14, bottom.

1. Written 1/3 1/15.

2. Certainly a concrete object. Influenced by the derivative *śpdt*, 'triangle', Moller (op. cit., No. 567) and Gardiner doubtfully (*Grammar*, p. 522) have classed the sign with 'Geometrical Figures'.

3. The triangles in the figures annexed to Problems 17 (see below) and 4 have the same form; see also the commentary below. The figures in Fig. 2.2 are facsimiles from the photographs.

Commentary

Here is a problem in regard to which there can fortunately be no possible doubt. A triangle of given area is such that (to use our terms) its perpendicular height is $2\frac{1}{2}$ times the base;[1] find both.

The measures employed are well known. The unit of area is the 'thousand-of-land', expressed here, as in the Kahun Papyri and often in the Rhind Papyrus, by a simple '1' for each 'thousand'; it was so called because it was regarded as made up of a thousand strips each 100 cubits (or one *khet*) long by 1 cubit broad.[2] It was equal to 10 *arurae* (or square *khet*), and thousands-of-land and decuple multiples of the *arurae* were used interchangeably in calculations, as we see from the odd-looking operation: twice 2 are 40.[3] The linear measure, although expressed merely by numerals, is obviously the *khet*.

We have seen in the preceding problem that, given the area of a rectangle and the proportions of its length and breadth, these latter dimensions can be ascertained. The Egyptians knew further that the area of a rectangle is double that of a triangle of the same 'length and breadth'.[4] Therefore a problem of this kind can be solved by doubling the given area of the triangle and then operating as in the preceding problem.

You are, then, *to double the area; result* 40 (arurae).[5] We now proceed as in Problem 6, with the difference that there we were told that the breadth was $\frac{3}{4}$ of the length, while here we are told that the length is $2\frac{1}{2}$ times the breadth.

A given square will have $2\frac{1}{2}$ times the area of a rectangle whose length is equal to a side of that square and is also $2\frac{1}{2}$ times as long as the adjacent sides. To find, then, the area of a square having its side equal to the length of our rectangle, *you are to take* the area of the latter $2\frac{1}{2}$ *times*; *result* 100. Of this 100 (*arurae*) you must now *take the square root*, *namely* 10, to find the side of the square,

1. The Egyptian expression 'the bank of $2\frac{1}{2}$' is dealt with below.

2. See Griffith in *Proc. Soc. Bibl. Arch.*, 1892, 415.

3. Similarly *Rhind Pap.*, No. 52: 'you are to multiply 20 (*arurae*) 5 times; result 10 (thousands-of-land).

4. See Peet, op. cit., 91 foll.; Gunn in *Journal*, xii, 133.

5. The italicized words actually occur in the Egyptian text.

equal to the length of our rectangle. You must now find the breadth of the latter; to do this, *evoke* 1 *from* multiples or fractions of $2\frac{1}{2}$, for the number or numbers which effect this will give us the number which has the same relation to 1 that 1 has to $2\frac{1}{2}$. We find that the fraction $\frac{1}{3}$ of $2\frac{1}{2}$ gives 2/3 1/6 (i.e. 5/6), and that 1/15 of $2\frac{1}{2}$ gives 1/6; since 5/6 and 1/6 are equal to 1, *what results as* the answer *is* $\frac{1}{3}+1/15$, or 2/5; i.e. 2/5 is to 1 as 1 is to $2\frac{1}{2}$. Now, as the length of our rectangle has been found to be 10, *apply* (the fraction 2/5) *to* 10, that is, take 2/5 of 10, to find the breadth; *result* 4. But we know that the rectangle thus obtained has the dimensions of the sought triangle; therefore the latter *is* 10 (*khet*) *in length by* 4 (*khet*) *in breadth.*

We might expect the solution to be followed, as in the preceding problem and in the very similar No. 17, by a figure and some sort of proof; but our scribe has not provided these.

PROBLEM NO. 17

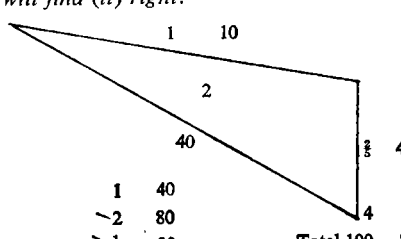
(*Transcription, Figure 2.2.*)

Example of calculating a triangle.

If you are told: A triangle of 2 *thousand* (*s-of-land*) *in its area, and what you put on the length, you must put* 2/5 *thereof*[1] *on the breadth:*

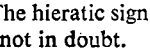
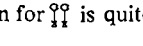
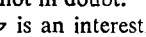
You are to double the 2 *thousands; result* 40 (arurae). *You are to treat* 2/5 *so as to find* 1*; result,* $2\frac{1}{2}$ *times. You are to take* 40 $2\frac{1}{2}$ *times; result* 100. *You are to calculate its square root; result* 10. *See, it is* 10 (khet) *in length. You are to take* 2/5 *of* 10*; result* 4. *See, it is* 4 (khet) *on the breadth.*

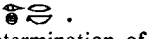
You will find (*it*) *right.*

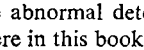


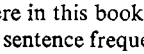
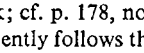
1. Here and below, 2/5 is a translation of the '1/3 1/15 'of the original.

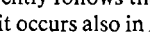
Notes on the Text

Lines 1, 2. The writing  (elsewhere in this book ) is interesting as showing the early reduction of *p* to *b* in this root, a change already known for the late period from  > coλτε and [> *śdbt*] > Σωθις; cf. Sethe in *Zeitschr. f. äg. Spr.*, L, 80.

Lines 2, 4. The hieratic sign for  is quite abnormally formed, but the reading is not in doubt.

Line 3.  is an interesting example of the Prospective Relative Form.

Line 7.  a bungle of  .

Line 8. The abnormal determination of a dimension-word by  occurs elsewhere in this book; cf. p. 178, note 4 below.

Line 9. This sentence frequently follows the solution of a problem in this book (cf. No. 14 below); it occurs also in *Berlin Pap.* 6619 (*Zeitschr. f. äg. Spr.*, XXXVIII, Pl. 4), No. 3.

Figure. In the original the triangle is carelessly drawn, with no regard to the proportions, as in the transcription. Three of the numerals to the left (in the original) of the triangle are damaged, but traces remain of all of them, and of the reading there is no doubt.

Commentary

In its nature and in the measures used,[1] this problem is similar to that of No. 7; the only difference is that here we are told that the breadth is 2/5 of the length, whereas in No. 7 we were told that the length was $2\frac{1}{2}$ times the breadth, statements which amount to the same thing, but which necessitate a variation in the Egyptian method of solution. As in No. 7, the given area of 2 thousands-of-land is doubled, giving 40 *arurae*, the equivalent of 4 thousands-of-land. The problem is now precisely similar to No. 6, and is worked out in the same way.

There follows a figure accompanied by some of the detailed working, not, as in No. 6, by the proof. In the triangle is a 2, indicating its area, 2 thousands-of-land. Beneath this is 40, the area in *arurae* of the rectangle formed by doubling the area of the

1. Note that the same dimensions occur in the one problem (No. 51) in the Rhind Papyrus dealing with the triangle, a problem which is the converse of the two dealt with above: 'If you are told: A triangle of 10 *khet* on its length (*mrit*) and 4 *khet* in its breadth (*tp-r*), what is its area? ..... (answer) its area is 2 (thousands-of-land).'

triangle. Below are the three lines needed to arrive at the product of this 40 and $2\frac{1}{2}$, namely 100, and to the last line is added 'square root 10'.[1] Over the triangle stands not simply its length 10 (*khet*) just found, but a 10 immediately preceded by a 1. It may be that the 1 marks the 10 as the object of multiplication in the following computation:

1	10
2/5	4

for to the left (in the original) of the 'breadth' is 2/5 4, standing for 2/5 of 10=4; but in such cases 1 as multiplier is always written in hieratic with a dot[2] instead of the usual stroke. Under the figures 2/5 4 stood the numeral 4; this is the measure in *khet* of the breadth, found by the above computation. As its correct place against the breadth was already filled up by the '2/5 4', the writer placed it as close as possible below.

The way in which the ratio of breadth to length of the triangle is stated in this problem is not without interest. In Problem 7 we saw that the ratio of length to breadth was expressed by the idiom '*ideb* of $2\frac{1}{2}$'. For the contrary ratio, however, namely breadth as fraction of length, there appears to be no technical term.

For our knowledge of the Middle Kingdom geometry of the triangle we now have four documents: *Rhind Pap.*, No. 51, *Moscow Pap.*, Nos. 7 and 17, discussed above, and No. 4, unpublished and incomplete, but of the same nature as *Rhind Pap.*, No. 51.[3] On these the following general remarks may be made:

1. They make it certain that the Egyptians of the period knew of the properties of the isosceles triangle which we should express by the following equations (a=area, h=height, b=base):

$$a=h.\frac{b}{2};\ h=\sqrt{2a.\frac{h}{b}};\ b=\frac{b}{h}\sqrt{2a.\frac{h}{b}}.$$

1. The lower part of this line is broken away; but the general dimensions of the papyrus show that there was no room for any further working-out.

2. Rendered, however, in our plates of transcription as a short stroke.

3. The obscure and faulty *Rhind Pap.*, No. 53 is disregarded here. In *Rhind Pap.*, No. 52 the translation, Peet, op. cit., 94–5, Gunn in *Journal*, XII, 133, ... 'truncated triangle' correctly renders the idea conveyed by the Egyptian word, but obscures the fact that we have to do, from our standpoint, not with any sort of triangle but with a symmetrical trapezoid.

2. There is nothing to show that these calculations were restricted to isosceles triangles. In the formation of the problems the general word *śpdt*, 'triangle', is used, and the solutions, both in the terms used and the methods employed, apply equally well to any sort of triangle. The figure attached to the Rhind problem is, it is true, isosceles; but that attached to *Moscow Pap.*, No. 17, is definitely scalene,[1] as is the determinative of the word *śpdt* in *Moscow Pap.*, Nos. 4, 7, 17. In the absence of any evidence to the contrary, we may take it as quite probable that the calculations summarized in the preceding paragraph were known to be valid for all triangles.

PROBLEM NO. 14

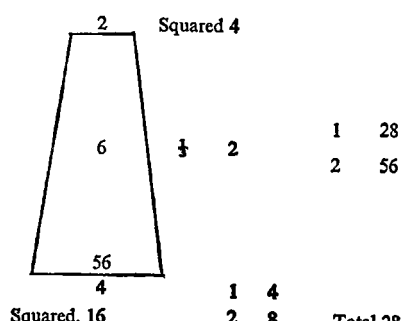
(*Transcription, Figure 2.3.*)

Example of calculating a truncated pyramid.

If you are told: A truncated pyramid of 6 *for the vertical height by* 4 *on the base by* 2 *on the top:*

You are to square this 4*; result* 16. *You are to double* 4*; result* 8. *You are to square this* 2*; result* 4. *You are to add the* 16 *and the* 8 *and the* 4*; result* 28. *You are to take* $\frac{1}{3}$ *of* 6*; result* 2. *You are to take* 28 *twice; result* 56. *See, it is of* 56.

You will find (*it*) *right.*



1. That of No. 4 is damaged, but as far as it goes it resembles that of No. 17.

Notes on the Text

The text is well preserved and presents no difficulties.

Figure. The solid is of course represented, as in the hieratic ideogram (see line 1), as a simple trapezoid, and in the original is roughly drawn without regard to the proportions, as in the transcription.

Commentary

The problem is to determine the volume of what we call a truncated pyramid, or frustum of a pyramid, the *data* being the vertical height (*stwti*) and the respective lengths of the sides of the two squares which bound the solid below and above.

If we call the height h and the sides of the lower and upper squares a and b respectively, the working may be represented as follows:

Square a, result 16. Multiply a by b, result 8. Square b, result 4. Add these results, total 28.

Take one-third of h, result 2. Multiply 28 by this, result 56, which is the volume sought.

Expressing these operations by a general formula, we have

$$V=(a^2+ab+b^2)\frac{h}{3},$$

which is exactly the formula used to-day to determine the volume of such solids.

The figure, and the numbers which accompany it, are quite straightforward. In the centre of the figure stands its height, 6. Below is the side a, namely 4, 'squared, 16', Similarly above we have the side b, namely 2, 'squared 4'. On the left (in the original), opposite the 6 inside the figure, we read $\frac{1}{3}$ 2, indicating that one-third of the height 6 is 2. Below on the left (in the original) is the multiplication of a and b, that is 4 multiplied by 2, and this is followed by the total, 28, of the 1 6, the 8 and the 4. Above this is the final step, the multiplication of 28 by 2, giving 56. This number, which is that of the required volume, is then inserted in the figure, near the base.

3

THE RHIND MATHEMATICAL PAPYRUS

(c. 1650 B.C.)

THE Rhind Mathematical Papyrus is the largest of the ancient Egyptian mathematical papyri extant and the earliest to be studied extensively. It was found at Thebes in a room of a ruined building near the Ramesseum. In 1858 it was purchased by the English archaeologist, A. Henry Rhind, after whom it is named. Through the agency of the British Museum, which purchased the papyrus from the Rhind estate in 1864, it has been made available in facsimile form for detailed study by scholars of Egyptology and mathematics. Originally a single scroll about eighteen and a half feet long and approximately thirteen inches high, it was received at first in two parts, a middle portion having been lost. Fragments of the missing portion were found in 1922 in a wholly unexpected place, among the medical papyri of the Edwin Smith collection in New York. The plates published in Professor T. Eric Peet's work on the *Rhind Mathematical Papyrus* (1923) now show the New York fragments in place, and the papyrus in virtually complete form.

The scroll containing the ancient mathematical treatise is made of papyrus three layers thick. The colour is a fine light brown and the hieratic script of the text is beautifully written in a strong, bold hand, the greatest part of it in black ink. The title, the first words of the sections and certain numbers are in red ink which still retains the brightness of its colour.

The scribe, whose name, A'h-mose (or Aah-mes, or Ahmes) means 'A'h (the moon-god)-is-born', stated that he had written this 'book' in the thirty-third year of the reign of the pharaoh 'A-user-Re, of the Hyksos dynasty (i.e. *c.* 1650 B.C.), and that it was a copy of an earlier work which had been written in the time of King Ne-ma'et-Re. The latter, known to us as King Amen-em-het III, was the ruler of Egypt from 1849–1801 B.C. Hence, the mathematical problems and processes of the Rhind represent methods which were in use during the period of the Middle Kingdom. Many of the problem types in the Rhind Papyrus are also to be found in the Moscow Papyrus and other papyrus fragments dating from 1850 B.C.

A chief feature of the ancient Egyptian mathematics is a highly original procedure for operating with fractions. The use of the Egyptian method of representing fractions survived to a very late date. The process of multiplication, restricted to doubling and selective addition, combined with a limitation in the representation of fractions to the use of unit fractions and 2/3, led to the need of a table of doubled unit fractions. This was given as a table of the division of 2 by odd numbers. The Egyptians used a decimal system of notation, they made frequent use of the *Regula falsi*, they verified their numerical answers, but had no procedure which was in the nature of a general proof. The geometrical problems were mensurational. Researches involving related Mesopotamian cultures confirm the view that the level and extent of Egyptian mathematics as shown in the Rhind Papyrus is representative of the mathematics of its period. The influence of Egyptian arithmetical methods can be observed in the later Greek papyri of the Hellenistic period.

THE RHIND MATHEMATICAL PAPYRUS

From The Rhind Mathematical Papyrus *by Thomas Eric Peet*

INTRODUCTORY

DESCRIPTION OF THE PAPYRUS

The *Rhind Papyrus* as it now lies in the British Museum consists of two pieces separately mounted between sheets of plate glass and numbered 10057 and 10058 respectively. These two pieces once formed a single roll, and were probably separated in modern times by an unskilful unroller.

The two sheets in the British Museum, Figure 3.1, may be described as follows:

Papyrus 10058

Present length 206 cm, height 33 cm

Recto. The recto consists of five pages of papyrus, each about 395 mm broad, except the first, and the last, which is incomplete. The gumming is admirably done. At the right-hand end there is

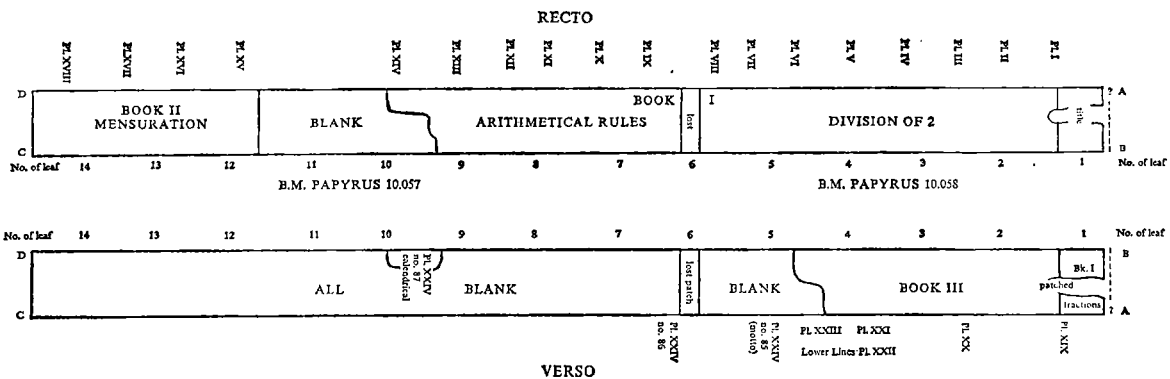


Fig. 3.1. Papyri 10058 and 10057. The plate numbers refer to the B.M. Facsimiles.

a blank space of about 10 cm, after which the title begins in vertical columns. This is followed by a double vertical black line, and from this point leftwards the sheet is ruled in black into six horizontal registers or bands throughout. The whole of this recto is devoted, with the exception of the title already referred to, to the division of 2 by the odd numbers from 3 to 101.

Verso. This face is very heavily patched, not only at the blank left-hand end, but also on the right: this patching was clearly done in ancient times, and where signs had disappeared on a lost fragment they have been written in in very black ink on the patches by a later hand. At 18 cm from the right-hand end is a double vertical ruling in black, as on the recto. Left of this the papyrus is ruled out into six horizontal registers. The writing begins at the same end as on the other face. On the right, outside the double line, is the carelessly written No. 61. It is followed by Nos. 62 to 84, and to the left of this last the papyrus is blank to its end (57 cm away), except for the curious No. 85, written upside down near the bottom and about half-way along the blank space.

Papyrus 10057

Present length 319 cm, height 33 cm or just over, the edge being actually under the framing.

Recto. The pages are from 39 to 40 cm broad, and the gumming is very accurate. The left-hand end is fragmentary. The whole face is ruled out in six horizontal registers. Problem No. 1 begins the recto and we run without a break up to No. 40; after which there is a blank of about 55 cm: No. 41 begins a fresh page, and the problems continue up to No. 60, which ends the recto.

Verso. This face is quite blank except for the calendrical entry No. 87, which is written near the top about halfway along. At the right-hand end the papyrus has been patched with a piece of another papyrus bearing the fragment of accounts numbered No. 86.

The following is a synopsis of the contents:
Table of resolution of fractions with numerator 2.

Book I. Arithmetic.

Nos. 1–6. Division of various numbers of loaves equally between 10 men.

Nos. 7–20. First group of completion-calculations (*sekem*) involving multiplication of fractions.

Nos. 21–3. Second group of completion-calculations, involving simple addition of fractions.

Nos. 24–34. Arithmetical solution by trial of equations of the first degree. (Eisenlohr's *hau*-calculations.)

Nos. 35–8. Similar equations involving the bushel or *hekat*.

Nos. 39 and 40. Division of loaves between men in unequal proportions.

Book II. Mensuration.

PART I. VOLUMES AND CUBIC CONTENT IN CORN.

Nos. 41–3. Cylindrical containers.

Nos. 44–6. Rectangular parallelopipedal containers.

No. 47. Expression in correct form of 1/10, 1/20, up to 1/100 of a *hekat*, disguised as a sum in cubic content.

PART II. AREAS

No. 48. Area of square and circle compared.

No. 49. Rectangle.

No. 50. Circle.

No. 51. Triangle.

No. 52. Truncated triangle.

No. 53. Trapezoid (?).

Nos. 54 and 55. Division of given area of land into equal-sized fields.

PART III. BATTER, OR ANGLE OF SLOPE.

Nos. 56–9. Batter of pyramid.

No. 60. Slope of a cone (?).

Book III. Miscellaneous problems in arithmetic.

No. 61. Multiplication of fractions (probably out of place, see above).

No. 62. Proportionate values of precious metals.

No. 63. Division of loaves in unequal proportions.
No. 64. Division of barley into shares in arithmetical progression.
No. 65. Division of loaves in unequal proportions.
No. 66. Daily portion of a yearly ration of fat.
No. 67. Reckoning of livestock.
No. 68. Division of 100 *hekat* of corn in unequal proportions.
Nos. 69–78. So-called *pefsu*-reckonings. Conversion of grain into bread and beer, and the barter of these last.
No. 79. Geometrical progression.
Nos. 80–1. Conversion of fractions of the *hekat* (1/2, 1/4, 1/8, etc.) into *henu*.
Nos. 82–3. Food estimate for a poultry yard.
No. 84. Estimate of food of an ox-stall.

Additions.

No. 85. Unintelligible group of signs.
No. 86. Fragment of accounts.
No. 87. Calendrical entries.

*

From The Rhind Mathematical Papyrus
Vol. I by Arnold B. Chace with the assistance of Henry P. Manning
Vol. II by Arnold B. Chace, Ludlow Bull and Henry P. Manning

FREE TRANSLATION AND COMMENTARY[1]

Title, Date and Scribe

Accurate reckoning. The entrance into the knowledge of all existing things and all obscure secrets. This book was copied in

1. In the following pages I have endeavoured to give in clear smooth English a free translation of the Rhind papyrus with some notes explaining details. In some places I have put in words or clauses that are omitted from the original, but found in other problems of the same group, or are required to express what I believe is the true meaning of the Egyptian. In some places where parts of a solution are misplaced I have arranged them in their proper order (see, for example, Problems 32 and 69). More particularly, I have corrected numerical mistakes and careless slips in the way of writing numbers. The reader will find the exact words of the original in the Literal

the year 33, in the fourth month of the inundation season, under the majesty of the king of Upper and Lower Egypt, 'A-user-Re', endowed with life, in likeness to writings of old made in the time of the king of Upper and Lower Egypt, Ne-mat'et-Re'. It is the scribe A'h-mose who copies this writing.

Title-page. Accurate reckoning[1] *of entering into things, knowledge of existing things all, mysteries . . . secrets all. Now was copied book this in year 33, month four of the inundation*[2]*-season* [*under the majesty of the*] *King of* [*Upper and*] *Lower Egypt,* '*A-user-Re*,[3] *endowed with life, in likeness to writings of old made in the time of the King of Upper* [*and Lower*] *Egypt,* [*Ne-ma*]'*et-*[*Re*'].[4] *Lo the scribe A'h-mose writes copy thus.*

Translation, and all corrections that have been made in the Free Translation may be found by a comparison of the two.

Notes giving explanations or interpretations of the text, except such as have been included in the Introduction, are placed in the following pages with the problems to which they refer. Notes relating to the Egyptian language or writing are placed with the Literal Translation.

1. Literally: 'head of reckoning', i.e. approved reckoning.

2. The *Akhet* is the first of the three seasons of the Egyptian year. It has been generally assumed that it was the season of inundation, partly because of the meaning of a word with a similar stem and partly because of the general belief that the Egyptian year originally began with the rise of the Nile. This view is set forth above, volume 1, page 44. However, the meanings of the names of the seasons are still uncertain, and it has been suggested that the New Year may originally have been placed in October, when the inundation begins to subside and the first quickly-growing crops are beginning to germinate, and that the *Shomu*, rather than the *Akhet*, was the season of the inundation.

3. In proper names compounded with divine names the divine name (in this case written with the sun-disk) is usually, as a mark of honour, placed first, whether it is to be read first or not. This name appears to mean 'Great-of-strength-is-Re'', but the syntax of the phrases and sentences often used as names by the ancient Egyptians, is not always clear.

4. This name may perhaps be rendered, 'He-who-belongs-to-the-truth-of-Re''. This is the so-called 'praenomen' or 'throne-name' of Amen-em-het III. The name of the scribe means, 'A'h (the moon-god)-is-born'.

EGYPTIAN ARITHMETIC

Table of the Division of 2 by Odd Numbers

2 divided by 3
Get 2 by operating on 3. $\frac{2}{3}$ of 3 is 2.

2 *divided by* 5
$\frac{1}{3}$ of 5 is $1\frac{2}{3}$, $\frac{1}{15}$ of 5 is $\frac{1}{3}$.

Working out:

1	5
$\frac{2}{3}$	$3\frac{1}{3}$
$1\backslash\frac{1}{3}$	$1\frac{2}{3}$
$\backslash\frac{1}{15}$	$\frac{1}{3}$.

2 divided by 7
$\frac{1}{4}$ of 7 is $1\frac{1}{2}\frac{1}{4}$, $\frac{1}{28}$ of 7 is $\frac{1}{4}$.

1	7		1	7
$\frac{1}{2}$	$3\frac{1}{2}$		2	14
$\backslash\frac{1}{4}$	$1\frac{1}{2}\frac{1}{4}$			
$2\backslash 4$	28	$\frac{1}{4}$	4	28

2 divided by 9
$\frac{1}{6}$ of 9 is $1\frac{1}{2}$, — $\frac{1}{18}$ of 9 is $\frac{1}{2}$.

1	9	
$\frac{2}{3}$	6	
$\frac{1}{3}$	3	
$\backslash\frac{1}{6}$	$1\frac{1}{2}$	
$\backslash 2$	18	$\frac{1}{2}$

2 divided by 11
$\frac{1}{6}$ of 11 is $1\frac{2}{3}\frac{1}{6}$, $\frac{1}{66}$ of 11 is $\frac{1}{6}$.

1	11		1	11	
$\frac{2}{3}$	$7\frac{1}{3}$		$\backslash 2$	22	
$\frac{1}{3}$	$3\frac{2}{3}$		$\backslash 4$	44	
$\backslash\frac{1}{6}$	$1\frac{2}{3}\frac{1}{6}$	Total	6	66	$\frac{1}{6}$.

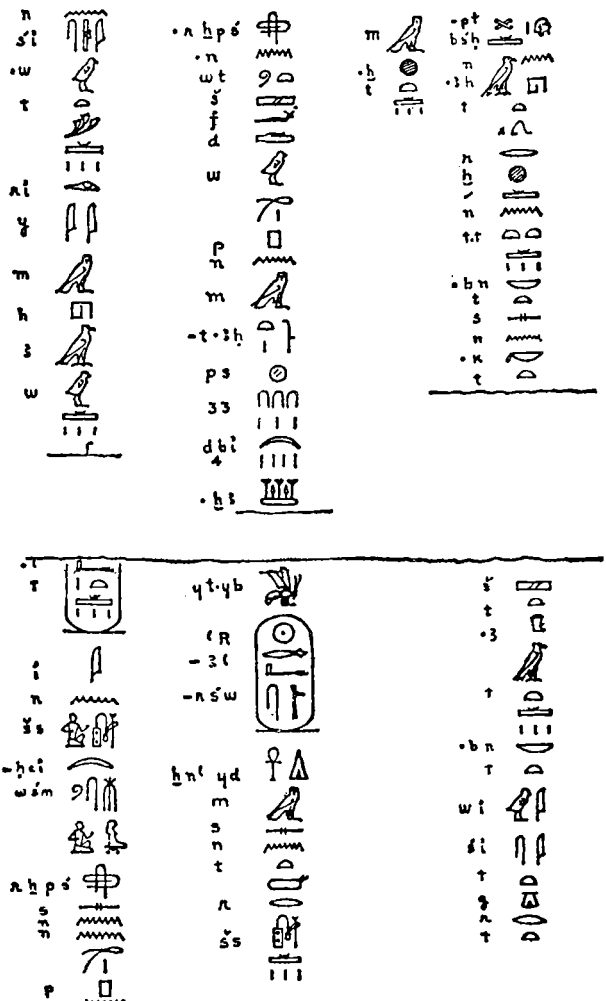


Fig. 3.2. Photograph 1, columns 1–3. B.M. Facsimile, Plate 1.

2 divided by 13

$\frac{1}{8}$ of 13 is $1\frac{1}{2}\frac{1}{8}$, $\frac{1}{52}$ of 13 is $\frac{1}{4}$, $\frac{1}{104}$ of 13 is $\frac{1}{8}$.

1	13	
$\frac{1}{2}$	$6\frac{1}{2}$	
$\frac{1}{4}$	$3\frac{1}{4}$	
$\frac{1}{8}$	$1\frac{1}{2}\frac{1}{8}$	
4	52	$\frac{1}{4}$
8	104	$\frac{1}{8}$.

2 divided by 15

$\frac{1}{10}$ of 15 is $1\frac{1}{2}$, $\frac{1}{30}$ of 15 is $\frac{1}{2}$.

1	15
$\frac{1}{10}$	$1\frac{1}{2}$
$\frac{1}{30}$	$\frac{1}{2}$.

2 divided by 17

Get 2 by operating on 17. $\frac{1}{12}$ of 17 is $1\frac{1}{3}\frac{1}{12}$, $\frac{1}{51}$ of 17 is $\frac{1}{3}$, $\frac{1}{68}$ of 17 is $\frac{1}{4}$.

Working out:

1	17		1	17	
$\frac{2}{3}$	$11\frac{1}{3}$		2	34	
$\frac{1}{3}$	$5\frac{2}{3}$				
$\frac{1}{6}$	$2\frac{1}{2}\frac{1}{3}$				
$\frac{1}{12}$	$1\frac{1}{4}\frac{1}{6}$	Total	3	51	$\frac{1}{3}$
Remainder	$\frac{1}{3}\frac{1}{4}$		4	68	$\frac{1}{4}$.

In the fourth line of the multiplication the author might have said $2\frac{2}{3}\frac{1}{6}$, and then the last line would have been $1\frac{1}{3}\frac{1}{12}$, and he uses this form, in giving his answer above, for the part that $\frac{1}{12}$ of 17 makes towards 2. Somewhat similarly in the case of 23 he changes $1\frac{1}{2}\frac{1}{4}\frac{1}{6}$ to $1\frac{2}{3}\frac{1}{4}$.

2 divided by 97

$\frac{1}{56}$ of 97 is $1\frac{1}{2}\frac{1}{8}\frac{1}{14}\frac{1}{28}$, $\frac{1}{679}$ of 97 is $\frac{1}{7}$, $\frac{1}{776}$ of 97 is $\frac{1}{8}$.

	1	97	
Find	$\frac{1}{56}$	$1\frac{1}{2}\frac{1}{8}\frac{1}{14}\frac{1}{28}$	
	7	679	$\frac{1}{7}$
	8	776	$\frac{1}{8}$.

2 divided by 99

$\frac{1}{66}$ of 99 is $1\frac{1}{2}$, $\frac{1}{198}$ of 99 is $\frac{1}{2}$.

	1	99	
Find	$\backslash\frac{2}{3}$	66	$1\frac{1}{2}$
	$\backslash 2$	198	$\frac{1}{2}$.

2 divided by 101

Get 2 by operating on 101. $\frac{1}{101}$ of 101 is 1, $\frac{1}{202}$ of 101 is $\frac{1}{2}$, $\frac{1}{303}$ of 101 is $\frac{1}{3}$, $\frac{1}{606}$ of 101 is $\frac{1}{6}$.

Working out:

$\backslash 1$	101	1
$\backslash 2$	202	$\frac{1}{2}$
$\backslash 3$	303	$\frac{1}{3}$
$\backslash 6$	606	$\frac{1}{6}$.

Table of the division of the numbers 1–9 by 10

Table	1	divided by 10	gives		$\frac{1}{10}$
	2	,,	,,	,,	$\frac{1}{5}$
	3	,,	,,	,,	$\frac{1}{5}$ $\frac{1}{10}$
	4	,,	,,	,,	$\frac{1}{3}$ $\frac{1}{15}$
	5	,,	,,	,,	$\frac{1}{2}$
	6	,,	,,	,,	$\frac{1}{2}$ $\frac{1}{10}$
	7	,,	,,	,,	$\frac{2}{3}$ $\frac{1}{30}$
	8	,,	,,	,,	$\frac{2}{3}$ $\frac{1}{10}$ $\frac{1}{30}$
	9	,,	,,	,,	$\frac{2}{3}$ $\frac{1}{5}$ $\frac{1}{30}$.

Problem 1. Example of dividing 1 loaf among 10 men.

Each man receives $\frac{1}{10}$.

Proof. Multiply $\frac{1}{10}$ by 10.

Do it thus:

1	$\frac{1}{10}$
$\backslash 2$	$\frac{1}{5}$
4	$\frac{1}{3}$ $\frac{1}{15}$
$\backslash 8$	$\frac{2}{3}$ $\frac{1}{10}$ $\frac{1}{30}$

Total 1 loaf, which is correct.

Problem 2. Divide 2 loaves among 10 men.
Each man receives $\frac{1}{5}$.
Proof. Multiply $\frac{1}{5}$ by 10.

Do it thus:	1	$\frac{1}{5}$
	\2	$\frac{1}{3}\ \frac{1}{15}$
	4	$\frac{2}{3}\ \frac{1}{10}\ \frac{1}{30}$
	\8	$1\ \frac{1}{3}\ \frac{1}{5}\ \frac{1}{15}$

Total 2 loaves, which is correct.

Problem 3. Divide 6 loaves among 10 men.
Each man receives $\frac{1}{2}\ \frac{1}{10}$.
Proof. Multiply $\frac{1}{2}\ \frac{1}{10}$ by 10.

Do it thus:	1	$\frac{1}{2}\ \frac{1}{10}$
	\2	$1\ \frac{1}{5}$
	4	$2\ \frac{1}{3}\ \frac{1}{15}$
	\8	$4\ \frac{2}{3}\ \frac{1}{10}\ \frac{1}{30}$

Total 6 loaves, which is correct.

Problem 4. Divide 7 loaves among 10 men.
Each man receives $\frac{2}{3}\ \frac{1}{30}$.
Proof. Multiply $\frac{2}{3}\ \frac{1}{30}$ by 10; the result is 7.

Do it thus:	1	$\frac{2}{3}\ \frac{1}{30}$
	\2	$1\ \frac{1}{3}\ \frac{1}{15}$
	4	$2\ \frac{2}{3}\ \frac{1}{10}\ \frac{1}{30}$
	\8	$5\ \frac{1}{2}\ \frac{1}{10}$

Total 7 loaves, which is correct.

Problem 22. Complete $\frac{2}{3}\ \frac{1}{30}$ to 1.
Applied to 30, $\frac{2}{3}\ \frac{1}{30}$ is 21. 30 exceeds 21 by 9. Multiply 30 so as to get 9.

1	30
\$\frac{1}{10}$	3
\$\frac{1}{5}$	6
Total	9.

Therefore $\frac{1}{5}\ \frac{1}{10}$ is to be added to make the completion.
For proof add them all together, namely,

$\frac{2}{3}\ \ \frac{1}{5}\ \ \frac{1}{10}\ \ \frac{1}{30}$, making 1;

for, applied to 30, these fractions are equal to

20 6 3 1, making 30.

Problem 24 (Figure 3.3). A quantity and its $\frac{1}{7}$ added together become 19. What is the quantity?

Assume 7.

\1	7
\$\frac{1}{7}$	1
Total	8.

As many times as 8 must be multiplied to give 19, so many times 7 must be multiplied to give the required number.

1	8
\2	16
$\frac{1}{2}$	4
\$\frac{1}{4}$	2
\$\frac{1}{8}$	1
Total $2\frac{1}{4}\frac{1}{8}$.	
\1	$2\frac{1}{4}\frac{1}{8}$
\2	$4\frac{1}{2}\frac{1}{4}$
\4	$9\frac{1}{2}$

Do it thus: The quantity is	$16\frac{1}{2}\frac{1}{8}$,
$\frac{1}{7}$	$2\frac{1}{4}\frac{1}{8}$,
Total	19.

Problem 40 (Figure 3.4). Divide 100 loaves among 5 men in such a way that the shares received shall be in arithmetical progression and that 1/7 of the sum of the largest three shares shall be equal to the sum of the smallest two. What is the difference of the shares?

Do it thus: Make the difference of the shares $5\frac{1}{2}$. Then the amounts that the 5 men receive will be

$$23 \quad 17\tfrac{1}{2} \quad 12 \quad 6\tfrac{1}{2} \quad 1, \text{ total } 60.$$

As many times as is necessary to multiply 60 to make 100, so many times must these terms be multiplied to make the true series.

\1	60
\$\frac{2}{3}$	40.

The total, $1\frac{2}{3}$, times 60 makes 100.

Multiply by $1\frac{2}{3}$

23	it becomes		$38\frac{1}{3}$
$17\frac{1}{2}$	,,	,,	$29\frac{1}{6}$
12	,,	,,	20
$6\frac{1}{2}$	,,	,,	$10\frac{2}{3}\frac{1}{6}$
1	,,	,,	$1\frac{2}{3}$
Total 60	,,	,,	100.

Problem 56 (Figure 3.5). If a pyramid is 250 cubits high and the side of its base 360 cubits long, what is its seked*?*

Take 1/2 of 360; it makes 180. Multiply 250 so as to get 180; it makes 1/2 1/5 1/50 of a cubit. A cubit is 7 palms. Multiply 7 by 1/2 1/5 1/50.

1	7
$\frac{1}{2}$	$3\frac{1}{2}$
$\frac{1}{5}$	$1\frac{1}{3}$ $\frac{1}{15}$
$\frac{1}{50}$	$\frac{1}{10}$ $\frac{1}{25}$

The *seked* is $5\frac{1}{2}\frac{1}{5}$ palms.

Problem 57. If the seked *of a pyramid is 5 palms 1 finger per cubit and the side of its base 140 cubits, what is its altitude?*

Divide 1 cubit by the *seked* doubled, which is 10–1/2. Multiply 10–1/2 so as to get 7, for this is a cubit: 7 is 2/3 of 10–1/2. Operate on 140, which is the side of the base: 2/3 of 140 is 93–1/3. This is the altitude.

In this inverse problem . . . the author doubles the *seked* instead of taking 1/2 of the side of the base, and instead of dividing the *seked* doubled by 7 and dividing the side of the base by the result, he divides 7 by the *seked* doubled and multiplies the side of the base by the result, which amounts to the same thing.

Problem 58. If a pyramid is 93–1/3 cubits high and the side of its base 140 cubits long, what is its seked*?*

Take 1/2 of 140, which is 70. Multiply 93–1/3 so as to get 70. 1/2 is 46–2/3, 1/4 is 23–1/3. Make thou 1/2 1/4 of a cubit. Multiply 7 by 1/2 1/4. 1/2 of 7 is 3–1/2, 1/4 is 1–1/2 1/4, together 5 palms 1 finger. This is its *seked.*

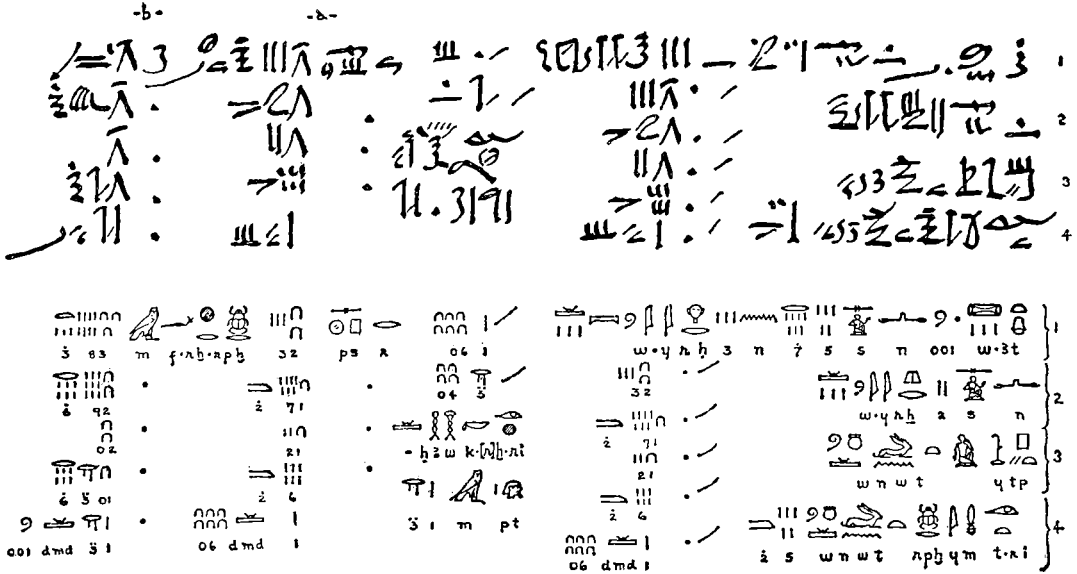


Fig. 3.4. Photographs XVI XVII, register 2. B.M. Facsimile, Plates XI–XII.

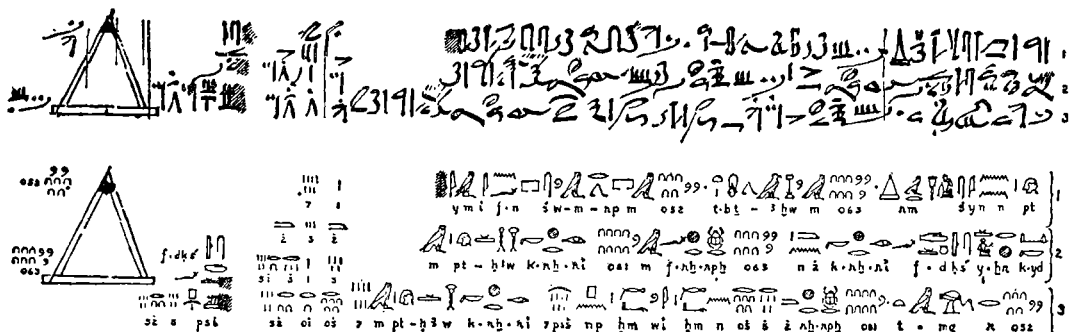


Fig. 3.5. Photographs XXI XXII, register 1. B.M. Facsimile, Plate XV.

The working out:

1	$93\text{-}\frac{1}{3}$
$\searrow\frac{1}{2}$	$46\text{-}\frac{2}{3}$
$\searrow\frac{1}{4}$	$23\text{-}\frac{1}{3}$
Total $\frac{1}{2}\ \frac{1}{4}$.	

Make thou 1/2 1/4 of a cubit; a cubit is 7 palms.

1	7
$\frac{1}{2}$	$3\ \frac{1}{2}$
$\frac{1}{4}$	$1\ \frac{1}{2}\ \frac{1}{4}$
Total	5 palms 1 finger

This is its *seked.*

Problem 61 (*Figure 3.6*). *Table for multiplication of fractions.*

$\frac{2}{3}$ of $\frac{2}{3}$ is $\frac{1}{3}\ \frac{1}{9}$
$\frac{1}{3}$,, $\frac{2}{3}$,, $\frac{1}{6}\ \frac{1}{18}$
$\frac{2}{3}$,, $\frac{1}{3}$,, $\frac{1}{6}\ \frac{1}{18}$
$\frac{2}{3}$,, $\frac{1}{6}$,, $\frac{1}{12}\ \frac{1}{36}$
$\frac{2}{3}$,, $\frac{1}{2}$,, $\frac{1}{3}$
$\frac{1}{3}$,, $\frac{1}{2}$,, $\frac{1}{6}$
$\frac{1}{6}$,, $\frac{1}{2}$,, $\frac{1}{12}$
$\frac{1}{12}$,, $\frac{1}{2}$,, $\frac{1}{24}$
$\frac{1}{9}$ of $\frac{2}{3}$ is $\frac{1}{18}\ \frac{1}{54}$; $\frac{1}{9}$, $\frac{2}{3}$ of it is $\frac{1}{18}\ \frac{1}{54}$
. . . . . .
$\frac{1}{5}$, $\frac{1}{4}$ of it is $\frac{1}{20}$
$\frac{1}{7}$, $\frac{2}{3}$,, ,, ,, $\frac{1}{14}\ \frac{1}{42}$
$\frac{1}{7}$, $\frac{1}{2}$,, ,, ,, $\frac{1}{14}$
$\frac{1}{11}$, $\frac{2}{3}$,, ,, ,, $\frac{1}{22}\ \frac{1}{66}$, $\frac{1}{3}$ of it is $\frac{1}{33}$
$\frac{1}{11}$, $\frac{1}{2}$,, ,, ,, $\frac{1}{22}$, $\frac{1}{4}$,, ,, ,, $\frac{1}{44}$

Peet points out . . . that the above table contains two forms of statement which have an interesting significance. In the first four lines we find forms of the type, 2/3 of 2/3 is 1/3 1/9, while in the last five are such forms as, 1/5, 1/4 of it is 1/20. In the ninth line both forms are given, the statement being made twice, while the four lines that immediately precede seem to have been originally in the second form but to have been changed to the first; see Literal Translation. The reason for the second form is that of the fractions in this table the only ones that were legitimate multipliers are 2/3 and 1/2 and fractions obtained

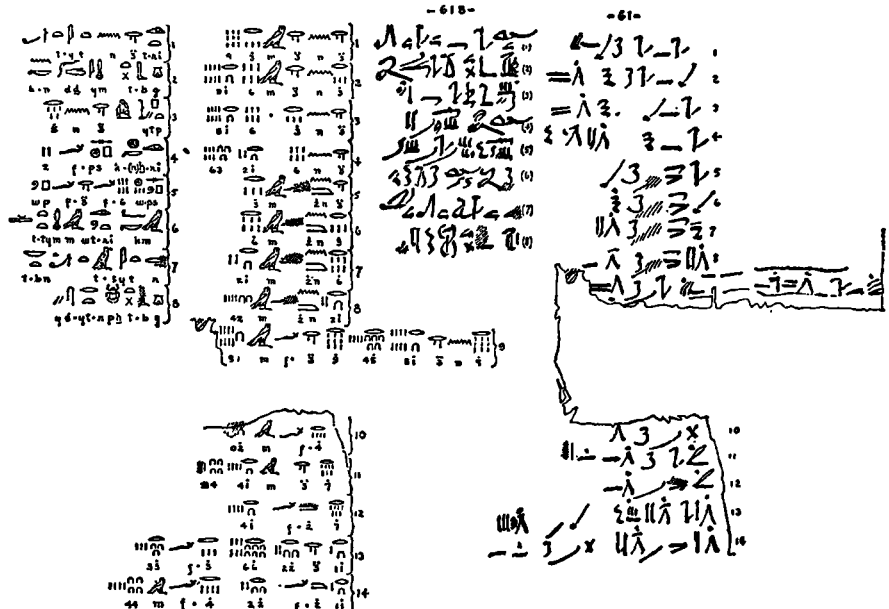


Fig. 3.6. Photograph XXIII, right half. B.M. Facsimile, Plate XVI.

from them by halving. Thus the Egyptian could not directly say, '1/5 of 1/4' but only '1/5, 1/4 of it'. He could have used the second form in all of these cases, but when he has a legitimate multiplier he prefers the first form, and so the author, perceiving after they were written that lines 5–8, like the first four, involve only these multipliers, changed them to the first form.

Problem 61B (*Figure 3.6*). *Rule for getting 2/3 of the reciprocal of an odd number.*

To get 2/3 of 1/5 take the reciprocals of 2 times 5 and 6 times 5, and in the same way get 2/3 of the reciprocal of any odd number.

Problem 62. Example of reckoning the contents of a bag of various precious metals. Suppose it is said to thee, A bag containing equal weights of gold, silver and lead, has been bought for 84 sha'ty. *What is the amount in it of each precious metal, that which is given for a* deben *of gold being 12* sha'ty, *for a* deben *of silver 6* sha'ty, *and for a* deben *of lead 3* sha'ty*?*

Add that which is given for a *deben* of each precious metal. The result is 21 *sha'ty*. Multiply 21 so as to get 84, the 84 *sha'ty* for which this bag was bought. The result is 4, which is the number of *deben* of each precious metal.

Do it thus:

Multiply 12 by 4 getting 48 *sha'ty* for the gold in the bag,
Multiply 6 by 4 getting 24 *sha'ty* for the silver,
Multiply 3 by 4 getting 12 *sha'ty* for the lead,
Multiply 21 by 4 getting 84 *sha'ty* altogether.

The *sha'ty* was a seal, and the word here represents a unit of value (see Weill, 1925). The *deben* was a unit of weight, equal to about 91 grammes. The papyrus does not say that the bag contains equal weights of gold, silver and lead, but in the solution the author proceeds as if this condition was understood.

Problem 79 (*Figure 3.7*). *Sum the geometrical progression of five terms, of which the first term is 7 and the multiplier 7.*

The sum according to the rule. Multiply 2801 by 7.

1	2801
2	5602
4	11204
Total	19607.

The sum by addition.

houses	7
cats	49
mice	343
spelt	2401
hekat	16807
Total	19607.

In the second column the author places before the successive powers of 7 the words, houses, cats, mice, spelt and *hekat*. Eisenlohr regards these words as names given to the powers of 7. Another interpretation is that the problem intended is like this: In each of 7 houses are 7 cats, each cat kills 7 mice, each mouse would have eaten 7 ears of spelt, and each ear of spelt will produce 7 *hekat* of grain; how much grain is thereby saved? But the author adds all of these quantities together, showing that he is more interested in the numerical problem of the sum of these numbers.

This interpretation, coupled with the number 7, reminds us of the children's rhyme, of which one version is the following:

'As I was going to Saint Ives,
I met a man with seven wives.
Every wife had seven sacks,
Every sack had seven cats,
Every cat had seven kits;
Kits, cats, sacks and wives,
How many were there going to Saint Ives?'[1]

Here, again, it is suggested that the sum total of a geometrical progression be calculated but there is a 'joker' in the actual wording of the first and last lines.

Rodet (1882, p. 111) found in the *Liber Abaci* of Leonardo of Pisa (see Bibliography, 1857) a problem of a geometrical progression expressed in much the same way, and having the ratio 7, and he suggests that Problem 79, absurd as is its heterogeneous addition, has perpetuated itself through all the centuries from the times of the ancient Egyptians.

1. *Every Child's Mother Goose*, with Introduction by Carolyn Wells. New York, 1918, p. 111.

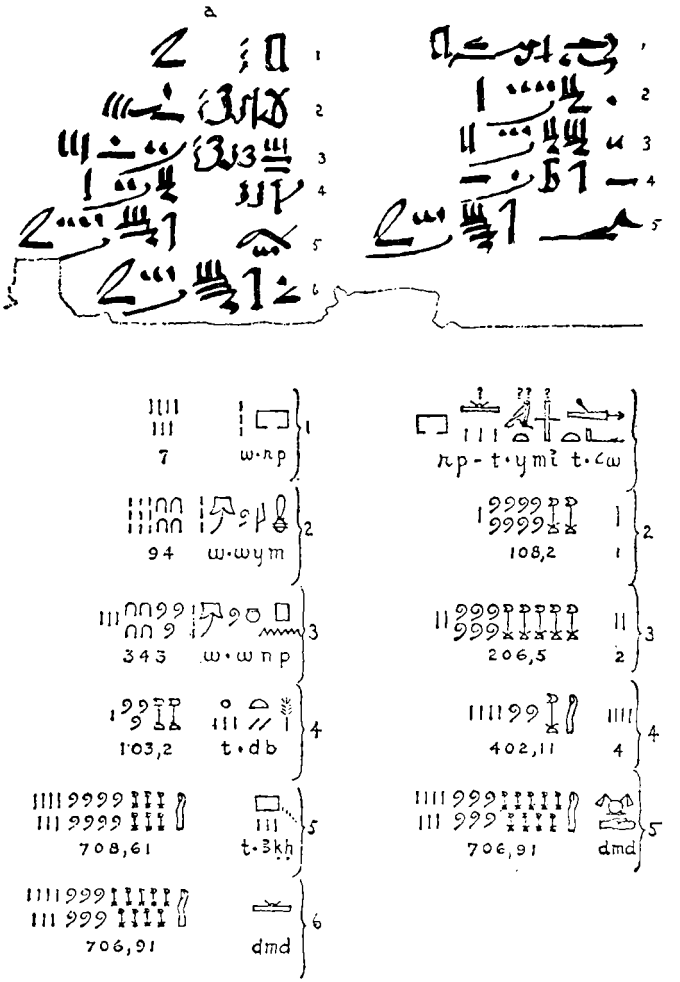


Fig. 3.7. Photographs XXV–XXVI, register 6. B.M. Facsimile, Plate XVIII.

Problem 80. Express the 'Horus eye' fractions in terms of the hînu.

The following vessels are used[1] in measuring grain by the functionaries of the granary:

1 *hekat*	makes	10		*hînu*
$\frac{1}{2}$,,	,,	5		,,
$\frac{1}{4}$,,	,,	$2\frac{1}{2}$		,,
$\frac{1}{8}$,,	,,	$1\frac{1}{4}$		,,
$\frac{1}{16}$,,	,,	$\frac{1}{2}\frac{1}{8}$		,,
$\frac{1}{32}$,,	,,	$\frac{1}{4}\frac{1}{16}$		,,
$\frac{1}{64}$,,	,,	$\frac{1}{8}\frac{1}{32}$		,,

Problem 81. Another reckoning. Express fractions of a hekat *as 'Horus eye' fractions and in terms of the* hînu.

Now $\frac{1}{2}$ *hekat*	makes	5		*hînu*
$\frac{1}{4}$,,	,,	$2\frac{1}{2}$		,,
$\frac{1}{8}$,,	,,	$1\frac{1}{4}$		,,
$\frac{1}{16}$,,	,,	$\frac{1}{2}\frac{1}{8}$		,,
$\frac{1}{32}$,,	,,	$\frac{1}{4}\frac{1}{16}$		,,
$\frac{1}{64}$,,	,,	$\frac{1}{8}\frac{1}{32}$		,,

a Now

$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$ *hekat*			makes	$8\frac{1}{2}\frac{1}{4}$ *hînu*							
$\frac{1}{2}$ $\frac{1}{4}$	,,		,,	$7\frac{1}{2}$	,,						
$\frac{1}{2}$ $\frac{1}{8}$ $\frac{1}{32}$	,,	$3\frac{1}{3}$ *ro*	,,	$6\frac{2}{3}$	,,	it is $\frac{2}{3}$	of a *hekat*				
$\frac{1}{2}$ $\frac{1}{8}$	,,		,,	$6\frac{1}{4}$	,,	,, $\frac{1}{2}$ $\frac{1}{8}$		,,	,,	,,	
$\frac{1}{4}$ $\frac{1}{8}$	,,		,,	$3\frac{1}{2}\frac{1}{4}$	,,	,, ,, $\frac{1}{4}$ $\frac{1}{8}$		,,	,,	,,	
$\frac{1}{4}$ $\frac{1}{16}$ $\frac{1}{64}$	,,	$1\frac{2}{3}$,,	,,	$3\frac{1}{3}$	,,	,, ,, $\frac{1}{3}$		,,	,,	,,	
$\frac{1}{4}$	,,		,,	$2\frac{1}{2}$	,,	,, ,, $\frac{1}{4}$		,,	,,	,,	
$\frac{1}{8}$ $\frac{1}{16}$	,,	4 ,,	,,	2	,,	,, ,, $\frac{1}{5}$		,,	,,	,,	
$\frac{1}{8}$ $\frac{1}{32}$	,,	$3\frac{1}{3}$,,	,,	$1\frac{2}{3}$	,,	,, ,, $\frac{1}{6}$		,,	,,	,,	

b Now

$\frac{1}{8}$ $\frac{1}{16}$	,,	4 ,,	,,	2	,,	,, ,, $\frac{1}{5}$		,,	,,	,,	
$\frac{1}{16}$ $\frac{1}{32}$	,,	2 ,,	,,	1	,,	,, ,, $\frac{1}{10}$		,,	,,	,,	
$\frac{1}{32}$ $\frac{1}{64}$	,,	1 ,,	,,	$\frac{1}{2}$	,,	,, ,, $\frac{1}{20}$		,,	,,	,,	
$\frac{1}{64}$	,,	3 ,,	,,	$\frac{1}{4}$	,,	,, ,, $\frac{1}{40}$		,,	,,	,,	
$\frac{1}{16}$	,,	$1\frac{1}{3}$,,	,,	$\frac{2}{3}$	,,	,, ,, $\frac{1}{15}$		,,	,,	,,	

1. This interpretation is suggested by Gunn.

4

THE BAKHSHALI MANUSCRIPT

(Composed fourth Century A.D. or earlier)

IN the spring of 1881, in a village called Bakhshali, on the north-west border of India, a farmer, digging in a ruined enclosure, came upon an old manuscript written on some seventy leaves of birch-bark, lying between stones. The contents of the manuscript proved to be mathematical, ranging over topics in arithmetic, algebra and mensurational geometry. Written in an old form of Sanskrit, it presented rules, illustrative examples, solutions and verifications of the results obtained. In its partly preserved and partly mutilated form, it appeared to be an interior portion of a much larger work. Lacking the beginning and the end of the work, we know neither the name of the work nor its author. Indian authorities believe that it is a running commentary on an earlier work and not a treatise on mathematics in the true sense. Dr A. F. Rudolf Hoernle was the first to study, edit and translate the manuscript.

Most of the illustrative problems are of the type requiring the solution of linear equations or of an indeterminate equation of the second degree. The solutions, performed principally by the method of false position, display the writer's knowledge of average value, series and a considerable skill in operations with fractions. Consistent use is made throughout the work of the decimal system of numeration with place values, an especially noteworthy feature being the heavy dot which serves the purpose of a zero in the representation of numbers. The heavy dot, indicating an empty place, is also used as a 'place-holder' for the unknown quantity in algebraic problems. Calculations involve positive and negative numbers, the sign (⊦) is used to denote a negative number or subtraction, and multiplication is frequently indicated by juxtaposition. While the manuscript itself may have been written as late as the ninth century, its contents were composed no later than the fourth century A.D. The mathematics of the Bakhshali manuscript thus antedates all known Hindu mathematical treatises.

THE BAKHSHALI MANUSCRIPT

by Dr A. F. Rudolph Hoernle

The manuscript is written in Sarada characters of a rather ancient type, and on leaves of birch-bark which from age have become dry like tinder and extremely fragile. Unfortunately, probably through the careless handling of the finder, it is now in an excessively mutilated condition, both with regard to the size and the number of the leaves. Their present size is about 6 by $3\frac{1}{4}$ inches; their original size, however, must have been about 7 by $8\frac{1}{4}$ inches. This might have been presumed from the well-known fact that the old birch-bark manuscripts were always written on leaves of a squarish size. But I was enabled to determine the point by a curious fact. The mutilated leaf which contains a portion of the twenty-seventh *sutra* shows at top and bottom the remainders of two large square figures, such as are used in writing arithmetical notations. These, when completed, prove that the leaf in its original state must have measured approximately 7 by $8\frac{1}{4}$ inches. The number of the existing leaves is seventy. This can only be a small portion of the whole manuscript. For neither beginning nor end is preserved; nor are some leaves forthcoming which are specifically referred to in the existing fragments. From all appearances, it must have been a large work, perhaps divided into chapters or sections. The existing leaves include only the middle portion of the work or of a division of it. The earliest *sutra* that I have found is the ninth; the latest is the fifty-seventh. The lateral margins which usually exhibit the numbering of the leaves are broken off. It is thus impossible even to guess what the original number of the leaves may have been.

The leaves of the manuscript, when received by me, were found to be in great confusion. Considering that of each leaf the top and bottom (nearly two-thirds of the whole leaf) are lost, thus destroying their connexion with one another, it may be imagined that it was no easy task to read the fragments and arrange them in order. After much trouble I have read and transcribed the whole, and have even succeeded in arranging in consecutive order

a not inconsiderable portion of the leaves containing eighteen *sutras*. The latter portion I have also translated into English.

The subject-matter is divided in *sutras*. In each *sutra* the matter is arranged as follows: First comes the rule, and then the example introduced by the word *udaharana*.[1] Next, the example is repeated in the form of a notation in figures, which is called *sthapana*. This is followed by the solution which is called *karana*. Finally comes the proof, called *pratyaya*.

The system of notation used in the Bakhshali arithmetic is much the same as that employed in the arithmetical works of Brahmagupta and Bhaskara.[2] There is, however, a very important exception. The sign for the negative quantity is a cross (+). It looks exactly like our modern sign for the positive quantity, but it is placed after the number which it qualifies. Thus $\begin{matrix}12\ 7+\\ 1\ \ 1\end{matrix}$ means $12-7$ (i.e. 5). This is a sign which I have not met with in any other Indian arithmetic; nor, so far as I have been able to ascertain, is it now known in India at all. The sign now used is a dot placed over the number to which it refers. Here, therefore, there appears to be a mark of great antiquity.

The following statement, from the first example of the twenty-fifth *sutra*, affords a good example of the system of notation employed in the Bakhshali arithmetic:

●	1	1	1	*bha* 32	*phalam* 108
1	1	1	1		
	3+	3+	3+		

Here the initial dot is used very much in the same way as we use the letter x to denote the unknown quantity the value of which is sought. The number 1 under the dot is the sign of the whole (in this case, unknown) number. A fraction is denoted by placing one

1. This word is almost uniformly abbreviated *uda*. Owing to the graphic symbols for *u* and *ta* being indistinguishable, I at first took the word to be complete and read it *tada*. But quite lately I found on a fragment, which had hitherto escaped my notice, the word written in full *udaharana*.

2. See Colebrooke's *Dissertation on the Algebra of the Hindus*. in his *Essays*, Vol. II, pp. 387 ff.

number under the other without any line of separation; thus $\frac{1}{3}$ is $\frac{1}{3}$, i.e. one-third. A mixed number is shown by placing the three numbers under one another; thus $\begin{matrix}1\\3\end{matrix}$ is $1+\frac{1}{3}$ or $1\frac{1}{3}$, i.e. one and one-third. Hence $\begin{matrix}1\\3+\end{matrix}$ means $1-\frac{1}{3}$ (i.e. $\frac{2}{3}$). Multiplication is usually indicated by placing the numbers side by side; thus

5	32	
8	1	*phalam* 20,

means $\frac{5}{8}\times 32=20$. Similarly $\begin{matrix}1&1&1\\3+&3+&3+\end{matrix}$ means $\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}$ or $(\frac{2}{3})^3$, i.e. ${}_2\frac{8}{7}$. *Bha* is an abbreviation of *bhaga*, 'part', and means that the number preceding it is to be treated as a denominator. Hence $\begin{matrix}1&1&1\\3+&3+&3+\end{matrix}$ *bha* means $1:{}_2\frac{8}{7}$ or $\frac{2}{8}\frac{7}{}$. The whole statement, therefore means $\frac{2}{8}\frac{7}{}\times 32=108$, and may

●	1	1	1		
1	1	1	1	*bha* 32	*phalam* 108
	3+	3+	3+		

be thus explained – 'a certain number is found by dividing with 8/27 and multiplying with 32; that number is 108'.

The dot is also used for another purpose, namely as one of the ten fundamental figures of the decimal system of notation, or the zero (0 1 2 3 4 5 6 7 8 9). It is still so used in India for both purposes, to indicate the unknown quantity as well as the nought. With us the dot, or rather its substitute the circle (0), has only retained the latter of its two intents, being simply the zero figure, or the 'mark of position' in the decimal system. The Indian usage, however, seems to show how the zero arose, and that it arose in India. The Indian dot, unlike our modern zero, is not properly a numerical figure at all. It is simply a sign to indicate an empty place or a hiatus. This is clearly shown by its name *sunya* 'empty'.

The empty place in an arithmetical statement might or might not be capable of being filled up, according to circumstances. Occurring in a row of figures arranged decimally or according to the 'value of position', the empty place could not be filled up, and the dot therefore signified 'nought', or stood in the place of the zero. Thus the two figures 3 and 7, placed in juxtaposition (37) mean 'thirty-seven', but with an 'empty space' interposed between them (3 7), they mean 'three hundred and seven'. To prevent misunderstanding the presence of the 'empty space' was indicated by a dot (3 • 7); or by what is now the zero (307). On the other hand, occurring in the statement of a problem, the 'empty place' could be filled up, and here the dot which marked its presence signified a 'something' which was to be discovered and to be put in the empty place. In the course of time, and out of India, the latter signification of the dot was discarded; and the dot thus became simply the sign for 'nought' or the zero, and assumed the value of a proper figure of the decimal system of notation, being the 'mark of position'. In its double signification, which still survives in India, we can still discern an indication of that country as its birth-place.

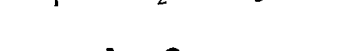
Generally speaking, the terms of an operation are set down side by side; and the particular operation intended is indicated by the initial syllable of a word of that import, subjoined to the terms which compose it. The operation of multiplication alone is not indicated by any special sign. Addition is indicated by *yu* (for *yuta*), subtraction by + (*ka* for *kanita*?) and division by *bha* (for *bhaga*). The whole operation is commonly enclosed between lines (or sometimes double lines), and the result is set down outside, introduced by *pha* (for *phala*). Occasionally the indicatory word is written in full. Vertical lines are usually interposed between the terms of a proportion or a progression. Thus:

$\begin{vmatrix} 5 & 7 \\ 1 & 1 \end{vmatrix}$ *yu*	*pha* 12 means	$5+7=12$
$\begin{vmatrix} 12 & 7+ \\ 1 & 1 \end{vmatrix}$	*pha* 5 „	$12-7=5$
$\begin{vmatrix} 5 & 32 \\ 8 & 1 \end{vmatrix}$	*pha* 20 „	$\frac{5}{8}\times 32=20$

● 1 1 1 1 1 1 1 *bha* 32 3+3+3+	*pha* 108	„	$(1 \div \frac{8}{27}) \times 32 = 108$
10 | 30 | 4 1 | 1 | 1	*pha* 12 1	„	10 : 30 = 4 : 12

Regarding the age of the manuscript, I am unable to offer a very definite opinion. The composition of a Hindu work on arithmetic, such as that contained in the Bakhshali MS., seems necessarily to presuppose a country and a period in which Hindu civilization and Brahmanical learning flourished. Now the country in which Bakhshali lies and which formed part of the Hindu kingdom of Kabul was early lost to Hindu civilization through the conquests of the Mohammedan rulers of Ghazni, and especially through the celebrated expeditions of Mahmud, towards the end of the tenth and the beginning of the eleventh centuries A.D. In those troublous times it was a common practice for the learned Hindus to bury their manuscript treasures. Possibly the Bakhshali MS. may be one of these. In any case it cannot well be placed much later than the tenth century A.D. It is quite possible that it may be somewhat older. The Sarada characters used in it, exhibit in several respects a rather archaic type, and afford some ground for thinking that the manuscript may perhaps go back to the eighth or ninth century. But in the present state of our epigraphical knowledge, arguments of this kind are always somewhat hazardous. The usual form in which the numeral figures occur in the manuscript are the following:

1 2 3 4



5 6 7 8 9 0

Quite distinct from the question of the age of the manuscript, is that of the age of the work contained in it. There is every reason to believe that the Bakhshali arithmetic is of a very considerably earlier date than the manuscript in which it has come down to us.

I am disposed to believe that the composition of the former must be referred to the earliest centuries of our era, and that it may date from the third or fourth century A.D. The arguments making for this conclusion are briefly the following:

In the first place, it appears that the earliest mathematical works of the Hindus were written in the *sloka* measure;[1] but from about the end of the fifth century A.D. it became the fashion to use the *arya* measure. Āryabhata *c.* A.D. 500, Varahamihira *c.* 550, Brahmagupta *c.* 630, all wrote in the latter measure. Not only were new works written in it, but also *sloka*-works were revised and recast in it. Now the Bakhshali arithmetic is written in the *sloka* measure; and this circumstance carries its composition back to a time anterior to that change of literary fashion in the fifth century A.D.

In the second place the Bakshali arithmetic is written in that peculiar language which used to be called the Gatha dialect, but which is rather the literary form of the ancient North-Western Prakrit (or Pali). It exhibits a strange mixture of what we should now call Sanskrit and Prakrit forms. As shown by the inscriptions (e.g. of the Indo-Scythian kings in Mathura) of that period, it appears to have been in general use, in North-Western India, for literary purposes till about the end of the third century A.D., when the proper Sanskrit, hitherto the language of the Brahmanic schools, gradually came into general use also for secular compositions. The older literary language may have lingered on some time longer among the Buddhists and Jains, but this would only have been so in the case of religious, not of secular, compositions. Its use, therefore, in the Bakhshali arithmetic points to a date not later than the third or fourth century A.D. for the composition of that work.

In the third place, in several examples, the two words *dinara* and *dramma* occur as denominations of money. These words are the Indian forms of the Latin *denarius* and the Greek *drakhme*. The former, as current in India, was a gold coin, the latter a silver coin. Golden *denarii* were first coined at Rome in 207 B.C. The Indian gold pieces, corresponding in weight to the Roman

1. See Professor Kern's Introduction to the *Brihat Samhita* of Varahamihira.

gold *denarius*, were those coined by the Indo-Scythian kings, whose line, beginning with Kadphises, about the middle of the first century B.C., probably extended to about the end of the third century A.D. Roman gold *denarii* themselves, as shown by the numerous finds, were by no means uncommon in India, in the earliest centuries of our era. The gold *dinaras* most numerously found are those of the Indo-Scythian kings Kanishka and Huvishka, and of the Roman emperors Trajan, Hadrian and Antoninus Pius, all of whom reigned in the second century A.D. The way in which the two terms are used in the Bakhshali arithmetic seems to indicate that the gold *dinara* and the silver *dramma* formed the ordinary currency of the day. This circumstance again points to some time within the three first centuries of the Christian era as the date of its composition.

A fourth point, also indicative of antiquity, which I have already adverted to, is the peculiar use of the cross (+) as the sign of the negative quantity.

*

There is also a diffusedness in the mode of composition of the Bakhshali work which reminds one of the similar characteristic observed in Buddhist and Jain literature. All these circumstances put together seem to render it probable that in the Bakhshali MS. there has been preserved to us a fragment of an early Buddhist or Jain work on arithmetic (perhaps a portion of a larger work on astronomy), which may have been one of the sources from which the later Indian astronomers took their arithmetical information. These earlier sources, as we know, were written in the *sloka* measure, and when they belonged to the Buddhist or Jain literature, must have been composed in the ancient North-Western Prakrit. Both these points are characteristics of the Bakhshali work. I may add that one of the reasons why the earlier works were, as we are told by tradition, revised and re-written in the *arya* measure by later writers such as Brahmagupta, may have been that in their time the literary form ('Gatha dialect') of the North-Western Prakrit had come to be looked upon as a barbarous and ungrammatical jargon as compared with their own classical Sanskrit. In any case the Buddhist or

Jain character of the Bakhshali arithmetic would be a further mark of its high antiquity.

TRANSLATION

18th Sutra

Twice the difference of the two initial terms, divided by the difference of the (two) increments, and further augmented by one, shall be the time that determines the progression.

FIRST EXAMPLE

A person has an initial (speed) of two and an increment of three, another has an increment of two and an initial (speed) of three. Let it now be determined in what time the two persons will meet in their journey.

The statement is as follows:

No. 1, init. term 2, increment 3, period x.
No. 2, init. term 3, increment 2, period x.

Solution: 'the difference of the two initial terms' 2 and 3 is 1; the difference of the two increments 3 and 2 is 1; twice the difference of the initial terms 1 is 2, and this, divided by the difference of the increments 1, is 2/1, and augmented by 1, is 3/1; this is the period. In this time [3] they meet in their journey which is 15.

27th Sutra

Now I shall discuss the wastage (in the working) of gold, the rule about which is as follows:

Having multiplied severally the parts of gold with the wastage, let the total wastage be divided by the sum of the parts of gold. The result is the wastage of each part (of the whole mass) of gold.

FIRST EXAMPLE

Suvarnas numbering respectively one, two, three, four, are subject to a wastage of mashakas numbering respectively one, two, three, four. Irrespective of such wastage they suffer an equal distribution of wastage. (What is the latter?)

The statement is as follows:

Wastage	$-1, -2, -3, -4$ mashaka.
Gold	1, 2, 3, 4 suvarna.

Solution: 'Having multiplied severally the parts of gold with the wastage,' etc.; by multiplying with the wastage, the products 1, 4, 9, 16 are obtained; 'let the total wastage', its sum is 30; the sum of the parts of gold is 10; dividing with it, we obtain 3. (This is the wastage of each part, or the average wastage, of the whole mass of gold.)

(Proof by the rule of three is the following) – as the sum of gold 10 is to the total wastage of 30 mashakas, so the sum of gold 4 is to the wastage of 12 mashakas, etc.

SECOND EXAMPLE

There are suvarnas numbering one, two, three, four. There are thrown out the following mashakas; one-half, one-third, one-fourth, one-fifth. What is the (average) wastage (in the whole mass of gold)?

Statement:

Quantities of gold	1, 2, 3, 4 suvarna.
Wastage	1/2, 1/3, 1/4, 1/5 mashaka.

Solution: 'Having multiplied severally the parts of gold with the wastage,' the products may thus be stated : 1/2, 2/3, 3/4, 4/5. 'Let the total wastage be divided'; the division being directed to be made the total wastage is 163/60; dividing 'by the sum of the parts of gold'; here the sum of the parts of gold is 10: being divided by this, the result is 163/600. This is the wastage of each part of the whole mass of gold.

Proof may be made by the rule of three: as the sum of the parts of gold 10 is to the total wastage of 163/60 mashaka, so the sum of gold 4 is to the wastage of 163/150 mashaka, etc.

50th Sutra

(The *sutra* is lost, but can be partially restored from the solution, and may be thus translated: 'The sum of the additive and subtractive numbers is divided by an assumed number; the quotient,

lessened by the same number and halved, is squared and added to the subtractive number.)

EXAMPLE

Which number added to five is a square, that (same) number lessened by seven is a square. Which number is that? This is the question.

53rd Sutra

(Having found) the two fractions (indicative) of the daily earnings, divide by their difference what is given towards (producing) equal possessions. The quotient, being doubled, is the time (in which their possessions become equal).

FIRST EXAMPLE

Let one hired Pandit earn five in three days; another learned man earns six in five days. The first gives seven to the second from his earnings. Say, in what time, after having given it, their possessions become equal?

Statement: No. I, 5/3=earnings of 1 day; No. II, 6/5=earnings of 1 day; gift 7.

Solution: 'The difference of the daily earnings; the two fractions; their difference'; (here the daily earnings are 5/3 and 6/5; their difference is 7/15; the gift is 7; divided by the difference of the daily earnings 7/15, the result is 15; being doubled, it is 30; this is the time), in which their possessions become equal.

Proof may be made by the rule of three, 3 : 5=30: 50, and 5 : 6=30 : 36; 'the first gives seven to the second' 7, remainder 43; hence 43 and 43 are their equal possessions.

SECOND EXAMPLE

Two Rajputs are the servants of a king. The wages of one (of them) per day are two and one-sixth, of the other one and one-half. The first gives to the second ten *dinaras*. Calculate and tell me quickly, in what time there will be equality (in their possessions)?

Statement: daily wages 13/6 and 3/2; gift 10.

Solution: 'and difference of the daily earnings'; here (the daily earnings are 13/6 and 3/2; their difference is 2/3; the gift is 10; divided by the difference of the daily earnings 2/3, the result is 15; being doubled, it is 30. This is the time, in which their possessions become equal).

Proof by the rule of three, $1 : 13/6 = 30 : 65$; and $1 : 3/2 = 30 : 45$. The first gives 10 to the second; hence 55 and 55 are their equal possessions.

Sutra

(Only the first portion of this *sutra* is preserved; viz. 'put into the empty place the number 1 representing the desired quantity, and then make up the series of items'. The purport of this rule will be understood from the following examples.)

FIRST EXAMPLE

(Its purport is: B gives 2 times as much as A, C gives 3 times as much as B, D gives 4 times as much as C. Their total gift is 132 What is the gift of A?)

Statement: A gives x, B 2, C 3, D 4. Total 132.

Solution: 'Put 1 in the place of x; then form the series of items' 1, 2, 3×2, 4×6, multiplying these several rates, 1, 2, 6, 24, their total is 33; with it divide the given total thus 132/33; the resulting item is 4, and this is the gift of A. Hence the series of gifts is as follows: 4, 8, 24, 96, and the total gift is 132. This is calculated from the series of items, and hence the total of the items is one hundred and thirty-two.

(Here follows what appears to be intended as a modification of the same *sutra* since it is not specialized as a separate *sutra*. What remains of it, runs thus: 'the number 1 is put into the empty place, and then (the items) are successively multiplied.' The purport of the rule will be again understood from the example.)

SECOND EXAMPLE

(Its purport is: B possesses 2 times as much as A; C has 3 times as much as A and B together; D has 4 times as much as A, B and C together. Their total possessions are 300. What is the possession of A?)

Statement: A has x, B 2, C 3×3, D 4×12. Total 300.

Solution: 'The desired quantity is put in the empty place'; the desired quantity is 1; this is placed as the first number; then the successive multiplications are made, 1, 2, 9, 48. Their addition gives the sum of the rates 60; with this the given total is divided, thus 300/60; the result is 5, and this is the possession of A. With this by multiplication the several rates are obtained, thus, 5, 10, 45, 240. Thence the total of the items is calculated to be 300.

FOURTH EXAMPLE

(Its purport is: A possesses something and $1\frac{1}{2}$ in addition; B has 2 times as much as A and $2\frac{1}{2}$ in addition; C has 3 times as much as B and $3\frac{1}{2}$ in addition; D has 4 times as much as C and $4\frac{1}{2}$ in addition. Their total possessions are one hundred and forty-four and one half. What is the possession of A?)

Statement: A has $x+1\frac{1}{2}$, B $2+2\frac{1}{2}$, C $3+3\frac{1}{2}$, D $4+4\frac{1}{2}$. Total $144\frac{1}{2}$.

Solution: 'Having put one in the empty place,' thus $1\div1\frac{1}{2}$: 'the several additions and multiplications are then made': in making the additions and multiplications, let the proper order of calculation be observed, (hence by addition) 5/2; next comes multiplication; (here) multiply numerator with numerator and denominator with denominator, 10/2 (i.e. $2/1\times5/2$); two and one half are now added, thus 15/2: now comes the multiplication with the third number, or three (is multiplied) with seven and one half (i.e. $15/2=7\frac{1}{2}$), thus 45/2; three and one half are now added, thus 52/2; now multiply the number four with twenty-six (i.e. $52/2=26$); the result is 208/2; four and one half are now added, thus 217/2. The total of these rates is 289/2 which is the given total of the possessions. All the rest remains the same; (i.e. dividing the given total 289/2 by the sum of the rates 289/2 we obtain 1 as the value of x, hence the possessions of A, B, C, D are respectively 5/2, 15/2, 52/2 and 217/2, the same as the rates mentioned above).

SEVENTH EXAMPLE

(Its purport is: A has $1\frac{1}{2}$ *plus* a certain amount; B has $2\frac{1}{2}$ less than 2 times A; C has $3\frac{1}{2}$ less than 3 times A; D has $4\frac{1}{2}$ less than 4

times A. Their total possessions are 29/2. What is the possession of A?)

(The statement is wanting.)

Solution: 'Having put the number one in the empty place', the addition is made 5/2; twice the rate of A less five halves is 5/2; three times the rate of A, less seven halves, is 8/2; four times the rate of A, less nine halves, is 11/2. The series of these rates is as follows: 5/2, 5/2, 8/2, 11/2. The given total is 29/2. The sum of the rates is 29/2. Dividing the one by the other, 29.2/2.29. we obtain 1. Multiplying by this, the same amount is obtained (as the gift of A; viz. 5/2). The same is the case with the negative quantities (i.e. B $1 \times [(2 \times 5/2) - 5/2] = 5/2$; similarly C 8/2, D 11/2).

5

THE MAYA CIVILIZATION

THE Maya civilization, one of the greatest of the pre-Hispanic American cultures, reached the height of its development during its Great or Classical Period, from the fourth to the ninth centuries A.D. Extending over a large area in Central America, it roughly spanned the present British Honduras, the provinces of Mexico situated on the Yucatan Peninsula, the Republic of Guatemala, and the western portions of Honduras and El Salvador. The Maya people lived in settled communities. The great majority of them engaged in agriculture and hunting. Dominating the scattered settlements in which they lived were the urban religious and ceremonial centres. The religion of the Maya greatly influenced all their activities. It was at the basis of the evolution of their architecture and their sculpture. It was the inspiration for the invention of their writing, their mathematics and their extraordinary calendar system. Evidences of their skill in precise stone-work have been found in pyramidal structures, which were used as platforms for temples or for observation posts. With the debris of centuries removed from some of the numerous mounds marking the sites of the ancient cities, stone pyramids have been found buried one beneath another, the innermost one serving as the tomb of a priest or other person of great dignity. Elaborate ornamentation on stone or on stucco facings shows a distinctive artistic development. Judging by the size of the pyramids (a pyramid in the ancient Maya city of Tikal, in the Department of Peten, Guatemala, rises to 229 feet) great numbers of people were employed in their construction. Many more were undoubtedly attracted to the cities during the religious festivals. At times, religious rites involved the gruesome spectacle of human sacrifices, but in later years great progress was made towards the abandonment of this practice. The Maya people were very advanced in ceramic arts. They had textiles and they wore personal ornaments of carved jade, bone, shell and pyrite.

When the stabilizing forces which had unified the distant Maya cities weakened, the peace which they had maintained was replaced by war, pestilence and famine. To stem the ensuing decline, an alliance of three of the city-states, Uxmal, Chichen-Itza and Mayapan, in the League of

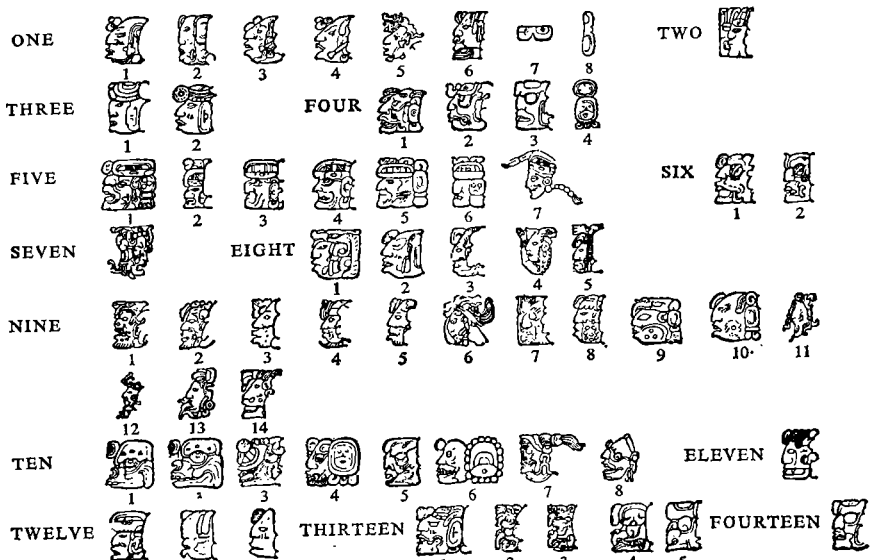
Mayapan, was formed (1004), and under the 'New Empire' religion, science and art flourished once more. However, in the succeeding centuries, the Maya were conquered by invaders from Mexico, first by the Toltecs and then by the Aztecs. In the mutual interchange of cultural influences between conqueror and conquered, the Maya feathered serpent god was accepted by the Aztecs and its representation became a prominent motif in Aztec art. It is believed, too, that the Aztecs took over the Maya calendar. Archaeological research has revealed that, long before the arrival of the Spaniards in Yucatan, there was a sudden abandonment by the Maya of one great city after another. The direct cause of the desertion has not yet been clearly established.

We possess little in the way of a documented history of the Maya. The slim sum total of such a history consists of reports by Spanish priests, a small number of native writings, and hieroglyphic inscriptions carved on the stone façades of buildings and upon shaftlike stone monuments known as 'stelae'. In his *Relacion de las cosas de Yucatan,* Bishop Diego de Landa tells of the burning of certain Maya books, written on long strips of bark or parchment, folded like a screen. Three of these books have survived and are now known as the 'codices'. They contain explanations and records of ceremonial rites, prophecies and historical events, written in hieroglyphs. The 'Books of Chilan (or Chilam) Balam' are also an important source of information. They were written with the encouragement of the Spaniards, by Maya writers in the Maya language, but in Spanish script. The paucity of documented history despite the possession by the Maya of a written language is due to the general destruction in the sixteenth century by the Spaniards, and by the natives who had been converted, of all articles which might serve to keep the old Maya traditions alive.

While much remains to be done in the decipherment of Maya inscriptions, the parts dealing with their numerical system and calendar are well understood. The Maya system of numeration employed place values, and it was the first in which systematic use was made of a symbol for zero. Maya mathematicians were skilful in computation. Their knowledge of astronomy combined with their arithmetic, enabled them to construct a complex calendrical system. The extremely accurate method of reckoning time, yielded by the Maya system, is all the more astonishing when it is remembered that it was fully developed and widely used as early as the fourth century A.D.

From The Numeration, Calendar Systems and Astronomical Knowledge of the Mayas, *by Charles P. Bowditch*

NUMBERS—FACE SIGNS



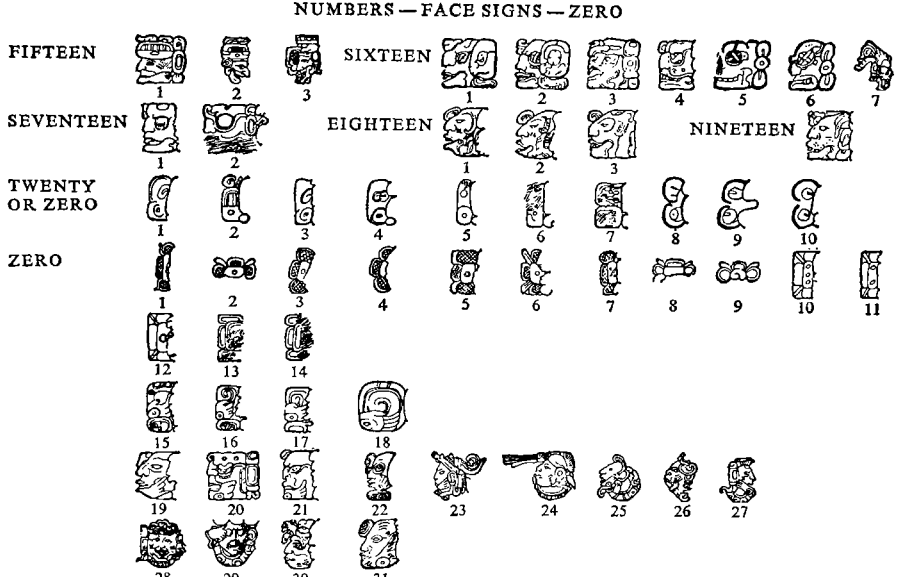


Fig. 5.2. Numbers – face signs – zero.

MAYA NUMERATION

CHARACTERISTICS OF HEAD-VARIANT NUMERALS 0 TO 19, INCLUSIVE

Forms	Characteristics
Head for 0	Clasped hand across lower part of face.
Head for 1	Forehead ornament composed of *more than one part*.
Head for 2	Oval in upper part of head. (?)
Head for 3	Banded headdress or fillet.
Head for 4	Bulging eye with square iris, snaglike front tooth, curling fang from back of mouth.
Head for 5	Normal form of tun sign as headdress.
Head for 6	'Hatchet eye'.
Head for 7	Large scroll passing under eye and curling up in front of forehead.
Head for 8	Forehead ornament composed of *one part*.
Head for 9	Dots on lower cheek or around mouth and in some cases beard.
Head for 10	Fleshless lower jaw and in some cases other death's head characteristics, truncated nose, etc.
Head for 11	Undetermined.
Head for 12	Undetermined; type of head known, however.
Head for 13	(*a*) Long pendulous nose, bulging eye, and curling fang from back of mouth. (*b*) Head for 3 with fleshless lower jaw of head for 10.
Head for 14	Head for 4 with fleshless lower jaw of head for 10.
Head for 15	Head for 5 with fleshless lower jaw of head for 10.
Head for 16	Head for 6 with fleshless lower jaw of head for 10.
Head for 17	Head for 7 with fleshless lower jaw of head for 10.
Head for 18	Head for 8 with fleshless lower jaw of head for 10.
Head for 19	Head for 9 with fleshless lower jaw of head for 10.

TABLE X. *From* An Introduction to the Study of Maya Hieroglyphs, *by Sylvanus G. Morley*

From The Numeration, Calendar Systems and Astronomical Knowledge of the Mayas, *by Charles P. Bowditch*

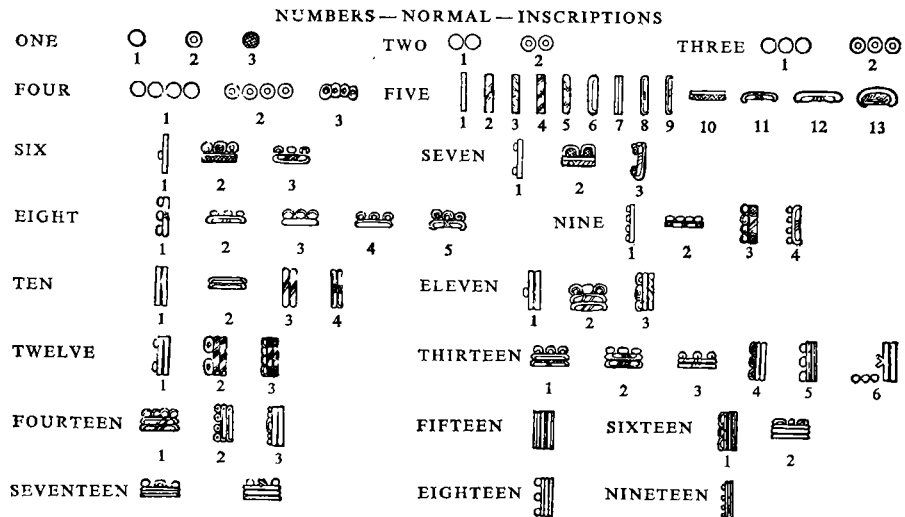


Fig. 5.3. Numbers – normal - inscriptions.

MAYA CALENDAR

From The Ancient Maya, *by Sylvanus Griswold Morley*

Since arithmetic and the calendar play such an important part in the Maya inscriptions, a brief description of Maya arithmetic and the Maya calendar is necessary to their better understanding.

TZOLKIN OR SACRED YEAR OF 260 DAYS

In all probability the only part of their highly elaborate calendar and chronology with which the common folk, the corn farmers, the hewers of wood and drawers of water, were familiar was the sacred year of 260 days, the *tzolkin* or 'count of days'. This time period was the most fundamental fact of their religion, since it determined for everybody the very pattern of his or her ceremonial life. The ancient Maya, man or woman, regarded his or her birthday not as the position in the tropical year, that is the month-day, upon which he was born, as we do, but the day of the *tzolkin*, or 260-day sacred year upon which he was born. The god of the particular day of this 260-day period upon which a man was born was his patron saint, his guardian deity, his celestial godfather, so to speak. Doubtless the god of the month in which he was born was also closer to him than the other eighteen month-gods, but his own particular *tzolkin* god was his strongest protector, his closest ally among the Heavenly Ones. Indeed, we have seen that among the Cakchiquels of the highlands of Guatemala a man took his given name from the day of the *tzolkin* upon which he happened to have been born, as Oxlahuh Tzii (Maya 13 Oc) for example.

The 260 days of the sacred year were formed by pre-fixing the numbers 1 to 13 inclusive to the twenty Maya day-glyphs, the names of which are given below, beginning with Ik, one of the Old Empire year bearers, and their corresponding hieroglyphs in Figure 5.4.

Ik	Manik	Eb	Caban
Akbal	Lamat	Ben	Eznab
Kan	Muluc	Ix	Cauac
Chicchan	Oc	Men	Ahau
Cimi	Chuen	Cib	Imix

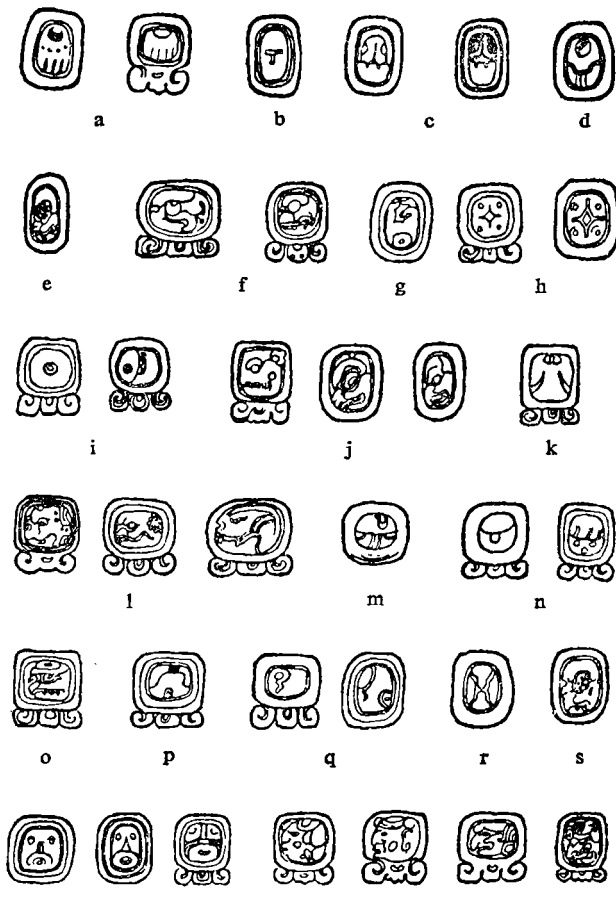


Fig. 5.4. Glyphs for the twenty Maya days: (*a*) Imix; (*b*) Ik; (*c*) Akbal; (*d*) Kan; (*e*) Chicchan; (*f*) Cimi; (*g*) Manik; (*h*) Lamat; (*i*) Muluc; (*j*) Oc; (*k*) Chuen; (*l*) Eb; (*m*) Ben; (*n*) Ix; (*o*) Men; (*p*) Cib; (*q*) Caban; (*r*) Eznab; (*s*) Cauac; (*t*) Ahau.

The Maya calendar, however, had no day named Ik, Akbal, or Kan alone, that is no day-name without an accompanying number, but, instead, each one of the twenty day-names in the list above, as already noted, had a number from 1 to 13 inclusive prefixed to it, thus – 1 Ik, 2 Akbal, 3 Kan, and so on. Not until every one of these thirteen numbers had been attached in turn to every one of the twenty day-names given above was a *tzolkin* complete.

If we commence with the number 1, for example, and prefix it to the first name in the list, Ik, viz. 1 Ik, and proceed without interruption to prefix the numbers from 1 to 13 inclusive in succession to each of these twenty names, the fourteenth name, Men, will have the number 1 again; the fifteenth name, Cib, the number 2 again; and so on until the first name, Ik in the second round of the twenty day-names comes back again with the number next in order, or 8. It follows, therefore, since 13 and 20 have no common factor, that, before any given number can be prefixed a second time to any given name, 259 days thereafter will have had to elapse, the 261st day having the same designation '1 Ik' as the first day, that is, one complete round of 260 differently named days – the *tzolkin* or 'count of days'. This is probably the oldest part of the Maya calendar, and certainly it was by far the most important to the masses, determining for each individual not only his birthday but also the ceremonial pattern of his life.

In order to give any day in the Maya calendar its complete description, however, it was further necessary to add to the days of the *tzolkin*, or 260-day ceremonial year, the corresponding month-position in the *haab* or 365-day calendar year which each occupied, as for example, 1 Imix 4 Uayeb, 2 Ik 0 Pop, 3 Akbal 1 Pop, 4 Kan 2 Pop, et cetera.

CALENDAR YEAR OF 365 DAYS

The Maya calendar year, or *haab*, was composed of 19 months – 18 months of 20 days each, and 1 closing month of 5 days, making a total of 365 month-positions in the calendar year. These

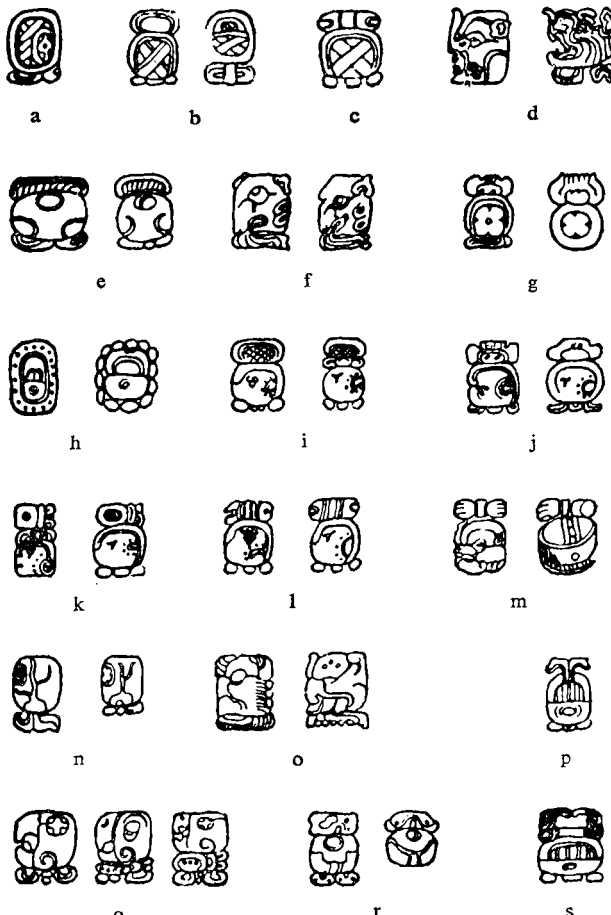


Fig. 5.5. Glyphs for the nineteen Maya months: (*a*) Pop; (*b*) Uo; (*c*) Zip; (*d*) Zotz; (*e*) Tzec; (*f*) Xul; (*g*) Yaxkin; (*h*) Mol; (*i*) Chen; (*j*) Yax; (*k*) Zac; (*l*) Ceh; (*m*) Mac; (*n*) Kankin; (*o*) Muan; (*p*) Pax; (*q*) Kayab; (*r*) Cumhu; (*s*) Uayeb.

19 divisions are given below and their corresponding hieroglyphs are shown in Figure 5.5.

Pop	Tzec	Chen	Mac	Kayab
Uo	Xul	Yax	Kankin	Cumhu
Zip	Yaxkin	Zac	Muan	Uayeb
Zotz	Mol	Ceh	Pax	

In order to show clearly how the 260 days of the *tzolkin* were combined with the 365 positions of the *haab* or calendar year, let us represent them graphically as two cogwheels (Figures 5,6), the smaller wheel, *A*, having 260 cogs, each named for one of the 260 days of the *tzolkin*, and the larger wheel, *B*, having 365 cogs, each intercog space being named for one of the 365 positions of the *haab* or calendar year.

Before we can enmesh these two wheels we must know two further facts about the Maya calendar. First, the Maya New Year's Day, or the first day of their first month, was written 0 (zero) Pop, Pop being the first month of the year (see above); and the first position in that month was written 0 rather than 1, as we would write it. This follows from the fact that the Maya regarded time as a series of elapsed periods, and a month position 1 Pop indicated to them the second day of Pop rather than the first day. This latter method is the way we number the hours of the day. For example, when we say one o'clock, in reality the first hour after noon is gone and we are about to start the second hour. So it was with the ancient Maya when they wrote 1 Pop; the first day, 0 Pop, had already passed and the second day (written 1 Pop) was about to commence. It follows from the foregoing that although the Maya months were each 20 days in length, except the last which had but 5 days, their month-positions were numbered from 0 to 19 inclusive, and in the case of the last month from 0 to 4 inclusive, and *not* as we would have numbered them from 1 to 20 inclusive and 1 to 5 inclusive in the case of the last month. This makes the first day of the Maya year 0 Pop, and not 1 Pop as we would write it.

The second fact we must know before combining the *tzolkin* with the *haab* is that only 52 of the 260 differently named days of the former could ever occupy the first position of the *haab*, or indeed, the first position of any one of its 19 divisions. These

52 days are those in which the names Ik, Manik, Eb and Caban appear (at the tops of the four columns earlier), and since each one of these names had the number 1 to 13 inclusive prefixed to it in turn, it results that only 52, or 4 × 13, of the 260 different days of the *tzolkin* could begin the Maya calendar year or any

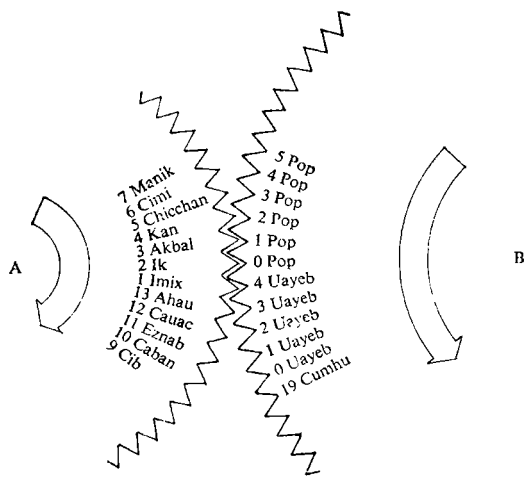


Fig. 5.6. Diagram showing the enmeshing of the 365-day calendar year (*B*) with the 260-day sacred year (*A*).

one of its 19 divisions. These 52 possible Maya New Year's Days, or year-bearers as they have been called, fell, during the Old Empire, upon the following days:

1 Ik	1 Manik	1 Eb	1 Caban	1 Ik
2 Manik	2 Eb	2 Caban	2 Ik	2 Manik
3 Eb	3 Caban	3 Ik	3 Manik	etc., etc.
4 Caban	4 Ik	4 Manik	4 Eb	
5 Ik	5 Manik	5 Eb	5 Caban	
6 Manik	6 Eb	6 Caban	6 Ik	
7 Eb	7 Caban	7 Ik	7 Manik	
8 Caban	8 Ik	8 Manik	8 Eb	
9 Ik	9 Manik	9 Eb	9 Caban	
10 Manik	10 Eb	10 Caban	10 Ik	
11 Eb	11 Caban	11 Ik	11 Manik	
12 Caban	12 Ik	12 Manik	12 Eb	
13 Ik	13 Manik	13 Eb	13 Caban	

It should be observed, however, that at the time of the Spanish Conquest, the days with which the Maya New Year began had shifted forward two positions, and that instead of beginning with days named Ik, Manik, Eb or Caban, they began with days named Kan, Muluc, Ix or Cauac (listed earlier).

CALENDAR ROUND

We are now in position to bring together, or enmesh, the two wheels in Figure 5.6, *A* representing the *tzolkin* and *B* the *haab* or calendar year. Let us do this in such a way that the cog of Wheel *A*, named after the day '2 Ik' of the *tzolkin*, will fit into the intercog-space on Wheel *B* corresponding to the Maya New Year's Day or position '0 Pop', giving the complete designation of this particular day as '2 Ik 0 Pop'.

Now let us revolve both wheels, *A* to the right like the hands of a clock, and *B* to the left, counterclockwise. Our problem is to find out how many complete revolutions each wheel will have to make before the cog named '2 Ik' on Wheel *A* will return to the intercog-space named '0 Pop' on Wheel *B*.

This problem is an old arithmetical friend of grammar-school days – the least common multiple. We must first ascertain the least common multiple of 260 and 365, during which process we shall also have ascertained how many complete revolutions the two wheels in Figure 5·6 will have to make before any one of the 260 different cogs of Wheel *A* will return to the same intercog-space of Wheel *B* from which it started, in a complete number of revolutions of each wheel.

Both 260 and 365 are divisible by 5, the first giving a quotient of 52, the second one of 73, but there is no further common factor, so the least common multiple of 260 and 365 is

$$5 \times 52 \times 73 = 18{,}980;$$

therefore our first Wheel, *A*, will make 73 complete revolutions, while the second Wheel, *B*, will make 52 complete revolutions before cog '2 Ik' of Wheel *A* will return to intercog-space 'O Pop' of Wheel *B* in a complete number of revolutions of each.

Translating this problem of arithmetic back into terms of the

Maya calendar, we may say that before any given day of the *tzolkin* could return to any given position of the *haab*, 73 *tzolkins* or 52 *haab*, or 18,980 days would have had to have elapsed. This is true since $73 \times 260 = 52 \times 365 = 18{,}980$ days.

In other words, once every 52 *haab* or calendar years of 365 days each, any given day, for example, '2 Ik', coincided with the first position of the year, '0 Pop', and this combination, or any other combination possible, recurred thereafter at intervals of 52 years each, throughout time. Thus any Maya who lived more than 52 years began to see New Year's Days of exactly the same name or, in fact, any other given days repeat themselves.

We do not know the ancient Maya name, or even the hieroglyph for this period of $52 \times 365 = 18{,}980$ days, important as it must have been, but modern students of the Maya calendar have called it the Calendar Round, because in this period of time the 260 days of the *tzolkin* will have occupied all of the 365 positions in the *haab* possible for them to occupy before the sequence will have begun to repeat itself, that is, one complete round of the 18,980 possible dates or 1 Calendar Round.

Not one of the peoples of Middle America who probably borrowed their respective calendars from the Maya, such as the Aztecs, the Mixtec and the Zapotec for example, ever devised or made use of any time period higher than this 18,980-day period (52 years of 365 days). The Aztecs, for example, conceived time as an endless succession of these 52 × 365 day-periods to which they gave the name *xiuhmolpilli*, meaning in their language 'year bundle', that is to say, the complete round of the years.

The Aztec had two special glyphs for this period, arising directly from their beliefs concerning it. The first was a knot (Figure 5.7*a*) indicating thereby that the bundle of 52 years had been tied up, finished or completed at the end of this period, and the second was the fire-drill and stick for kindling the Sacred Fire (Figure 5.7*b*). The Aztec believed that the world would come to an end only at the close of one of these 52-year periods; and on the last night of the *xiuhmolpilli*, we are told, the population of Tenochtitlan (Mexico City) withdrew to the hills surrounding the city to await the coming of dawn. When the sun rose on that morning, there was general rejoicing, the Sacred Fire was

rekindled, the houses were cleaned and set in order and the business of life was resumed as usual. The gods had given mankind another 52-year lease of life.

Fig. 5.7. Aztec glyphs for the *xiuhmolpilli* or 52-year period: (*a*) two examples of the knot; (*b*) two examples of the stick and drill for kindling the Sacred Fire.

Neither the Maya name nor the corresponding glyph for this highly important time period is known, although it was almost certainly a Maya conception originally, being the basis of their calendar system.

The Maya, like the Aztec, also conceived time as an endless succession of these 18,980 different possible combinations of the 260 days (the *tzolkin*) and the 365 different positions of the calendar year (*haab*); but at a very early date, probably as early as the fourth century before the beginning of the Christian Era, they perceived that even the first few multiples only of this period – 37,960, 56,940, 75,920, 94,900, 113,880, etc. – would involve them immediately in complex numbers, which became increasingly difficult to handle as time went on.

MAYA ARITHMETIC

In order to escape such rapidly mounting calendric chaos, the ancient Maya priests devised a simple numerical system which even today, more than two thousand years later, stands as one of the most brilliant achievements of the human mind.

Some time during the fourth or third centuries before Christ, the Maya priests *for the first time in the history of the human race devised a system of numeration by position, involving the*

conception and use of the mathematical quantity of zero, a tremendous abstract intellectual accomplishment.

Formerly it was believed, and not so many years ago either, that positional mathematics and the conception of zero which it involves had been developed but once in human history, namely, by the Hindus who devised the decimal notation with its accompanying numerical symbols about the eighth century of the Christian Era. From India this decimal numerical system passed to Arabia, hence the term Arabic numerals, and was carried from Arabia by the Arabs into Egypt. From northern Africa the Moors introduced it into Spain, and it did not come into general use among our own ancestors of western Europe until the fifteenth century – about seven hundred years after its invention in southern central Asia.

We now know, however, that the ancient Maya had developed their own system of positional mathematics, based upon 20 as its unit of progression instead of 10, that is, vigesimal instead of decimal, at least a thousand years earlier than its invention in the Old World by the Hindus and nearly two thousand years before positional mathematics came into general use among our own ancestors of western Europe.

But for a single break in the third order of units to make the third order approximate the length of the solar year as nearly as possible, the Maya vigesimal system is almost as simple as our own decimal system.

The unit of the Maya calendar was the day or *kin*. The second order of units, consisting of 20 *kins*, was called the *uinal*. In a perfect vigesimal system of numeration, the third term should be 400, that is, $20 \times 20 \times 1$, but at this point, *in counting time only*, the Maya introduced a single variation, in order to make the period of their third order agree in length with their calendar year as nearly as possible. The third order of the Maya vigesimal system, the *tun*, therefore, was composed of 18 (instead of 20) *uinals*, or 360 (instead of 400) *kins*; 360 days or *kins* being a much closer approximation to the length of the 365-day calendar year than 400 days.

Above the third order, however, the unit of progression used to form all the higher numbers is uniformly 20, as will be seen

below, where the names and numerical values of the nine known different orders of time periods are given:

20 *kins* = 1 *uinal* or 20 days
18 *uinals* = 1 *tun* or 360 days
20 *tuns* = 1 *katun* or 7,200 days
20 *katuns* = 1 *baktun*[1] or 144,000 days
20 *baktuns* = 1 *pictun* or 2,880,000 days
20 *pictuns* = 1 *calabtun* or 57,600,000 days
20 *calabtuns* = 1 *kinchiltun* or 1,152,000,000 days
20 *kinchiltuns* = 1 *alautun* or 23,040,000,000 days

The break in the third order of units mentioned above, 360 instead of 400, the latter of which is the correct value of the third term in a strictly vigesimal system, however, was used only *in counting time*; in counting everything else the Maya followed the vigesimal progression consistently : 1, 20, 400 (instead of 360), 8,000 (instead of 7,200), 160,000 (instead of 144,000), 3,200,000 (instead of 2,880,000), and so on.

TWO FORMS FOR EACH MAYA GLYPH

Practically speaking every Maya hieroglyph occurs in two forms in the inscriptions – (1) what has been called the normal or regular form and (2) a head-variant, the latter being the head of a deity, man, animal, bird, serpent, or even some mythological creature who lived only in the minds of his creators; and very, very rarely (only seven examples known), there is a third form where the glyph is the full figure of a deity, man, animal, bird or serpent.

The glyphs for the foregoing nine time-periods are given in Figure 5.8, normal forms at the left, head-variants at the right. In the cases of the last three periods, corresponding head-variants have not been identified as yet.

Like ourselves, the ancient Maya made use of two different notations in writing their numbers: (1) bar-and-dot numerals which may be compared to our own Roman notation, and (2)

1. The period of the fifth order, the *baktun*, was originally called the 'cycle' by modern investigators. The ancient name for this period, however, was probably *baktun* as given above.

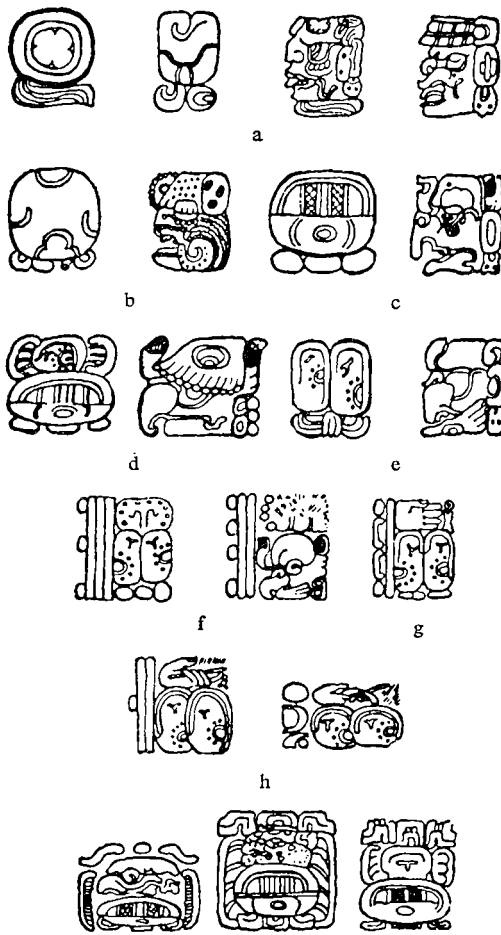


Fig. 5.8. Glyphs for the nine known Maya time-periods: (*a*) kin; (*b*) uinal; (*c*) tun; (*d*) katun; (*e*) baktun; (*f*) pictun; (*g*) calabtun; (*h*) kinchiltun; (*i*) alautun or Initial Series introducing-glyph.

head-variant numerals, which may be likened to our Arabic notation.

In the first notation, the dot • has a numerical value of I and the bar —— a numerical value of V, and by varying combinations of these two symbols, the numbers from I to XIX inclusive were written as shown in Figure 5.9 The numbers *above* XIX, however, involved the use of their positional mathematical system, already mentioned, and will be described later.

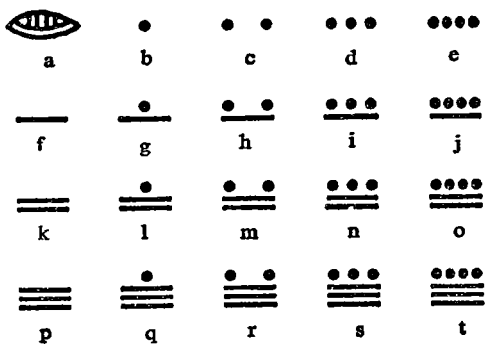


Fig. 5.9. Glyphs for the numbers 0 and I to XIX inclusive, in bar-and-dot notation, the Maya 'Roman Notation'. (*a*) zero; (*b*) I; (*c*) II; (*d*) III, (*e*) IV; (*f*) V; (*g*) VI; (*h*) VII; (*i*) VIII; (*j*) IX; (*k*) X; (*l*) XI; (*m*) XII; (*n*) XIII; (*o*) XIV; (*p*) XV; (*q*) XVI; (*r*) XVII; (*s*) XVIII; (*t*) XIX.

It is readily apparent from Figure 5.9 that the Maya bar-and-dot notation was superior to Roman notation in two respects. To write the numbers from I to XIX inclusive in Roman notation, it is necessary to employ three symbols – the letters I, V and X – and two arithmetical processes – addition and subtraction: VI is V plus I, but IV is V minus I.

On the other hand, in Maya bar-and-dot notation, in order to write the numbers from 1 to XIX inclusive, it is necessary to employ only two symbols – the dot and the bar – and only one arithmetical process, namely addition. In other words, Maya bar-and-dot notation used not only one symbol less to write the numbers from I to XIX inclusive than Roman notation requires, but also one arithmetical process less.

The second notation employed by the ancient Maya in writing their numbers made use of different types of human heads to represent the numbers from 1 to 13, inclusive, and zero. The Maya head-notation is comparable to our own Arabic notation, wherein there are ten different symbols representing zero and the first nine numbers – 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. These fourteen Maya head-variant numerals are nothing more nor less than the heads of the patron deities of the first fourteen numbers described in the preceding chapter.

It will be remembered that in forming the days of the *tzolkin*, 13 numbers from 1 to 13 inclusive were prefixed to the glyphs for the twenty days in succession. There is evidence that the ancient Maya regarded the first thirteen numbers and zero as *the primary numbers*, since each one of them has a special head to represent it, that is each of those numbers has its own patron deity.[1]

The head-variant for 10 is the death's head, or skull, and in forming the head-variants for the numbers from 14 to 19 inclusive the fleshless lower jaw of the death's head was the part used to represent the value of 10 in these composite heads for these six higher numbers. Thus if the fleshless lower jaw is applied to the lower part of the head for 6, which is characterized by a pair of crossed sticks in the large eye socket, the resulting head will be that for 16; that is, 10+6. Or again, if this same fleshless lower jaw is applied to the lower part of the head for 9, which is characterized by a circle of dots on the lower cheek, the resulting head will be that for 19; or 10+9; further, by applying the fleshless lower jaw to the heads for 4, 5, 7 and 8, the heads for 14, 15, 17 and 18, respectively, are also obtained.

It is not improbable that the 13 heads representing the 13 numbers from 1 to 13 inclusive are those of the *Oxlahuntiku* or Thirteen Gods of the Upper World as opposed to the *Bolontiku* or Nine Gods of the Lower World, and that each one of the former was associated with one of these thirteen numbers, being its especial patron.

1. The head-variant for the number 11 has not yet been surely identified.

MAYA VIGESIMAL SYSTEM OF MATHEMATICS

In order to write the numbers *above* the first order of units (those above 19), the ancient Maya made use of their positional system of numeration. In our own decimal system, the positions to the *left* of the decimal point *increase by tens from right to left* – units, tens, hundreds, thousands, etc. In the Maya positional system, however, the values of the positions *increase by twenties from bottom to top, with the single exception of the third position which, in counting time alone, is only* 18 *instead of* 20 *times the second,* an irregularity already mentioned.

To illustrate this, let us see how the Maya would have written various numbers of days above 20, selected at random. Take for example, the number 20 itself. This is 1 complete unit of the second order and no units of the first order, and thus involves two symbols, a symbol for zero in the first or lowest position to show that no units of the first order are involved in the number, and one unit of the second order.

One of the commonest symbols for zero in the Maya hieroglyphic writing was the conventionalized shell (the lower or first symbol in Figure 5.10*a*), and by placing a shell in the first or lowest position to denote 0 units of the first order and a dot • in the second position to denote 1 unit of the second order, the number 20 was written (Figure 5.10*a*).

The number 37 was written as shown in Figure 5.10*b*, that is, 17 units of the first order and 1 unit of the second order; the number 300 as shown in Figure 5.10*c*, that is, 0 units of the first order and 15 units of the second order. The number 360, the third order of units, was written as shown in Figure 5.10*d*, that is, 0 units of the first order, 0 units of the second order, and 1 unit of the third order of days.

The number 7,113 was written as shown in Figure 5.10*e*, that is, 13 units of the first order, 13 units of the second order, and 19 units of the third order. The number 7,202, involving four orders of time units, is shown in Figure 5.10*f*, or 2 units of the first order, 0 units of the second order, 0 units of the third order, and 1 unit of the fourth order. The number 100,932 is shown in Figure 5.10*g*, or 12 units of the first order, 6 units of the second order,

0 units of the third order and 14 units of the fourth order. Finally the number 169,200, involving five orders of time units, is shown in Figure 5.10*h*, or 0 units of the first order, 0 units of the second order, 10 units of the third order, 3 units of the fourth order, and 1 unit of the fifth order.

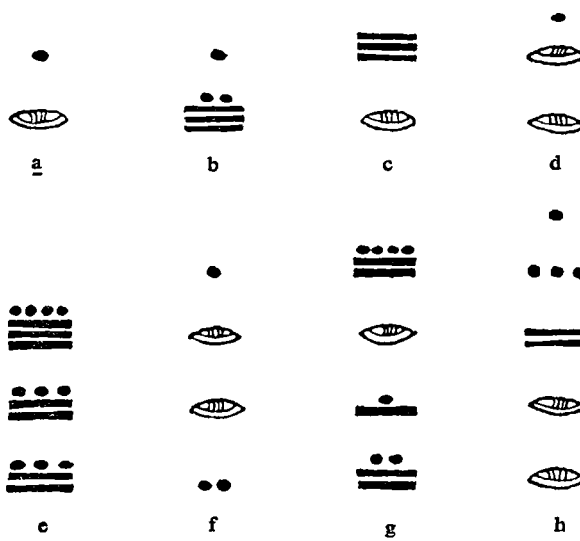


Fig. 5.10. Examples of Maya numbers above XIX in bar-and-dot notation: (*a*) 20; (*b*) 37; (*c*) 300; (*d*) 360; (*e*) 7,113; (*f*) 7,202; (*g*) 100,932; (*h*) 169,200.

MAYA CHRONOLOGY

STARTING POINT OF THE SYSTEM

In addition to this masterpiece of mathematical achievement, the ancient Maya seem to have been the first people anywhere in the world to have perceived the necessity for having a fixed base, or starting point, some definite event, either real or imaginary, from which to count their chronological era.

The ancient Egyptians dated their monuments and temples from the years of the reigns of the Pharaohs who erected them –

'In the third year of the reign of the Son of the Sun, Thutmose'; 'In the fifth year of the reign of the Son of the Sun, Rameses', etc. But this method of dating was relatively crude and inaccurate. If no monument or temple happened to have been erected during the closing years of any particular reign, these closing years dropped from the total of the record, and this error was always cumulative, never compensative, so that in the course of the several thousand years of the Pharaonic Period, this one source of error alone amounted to several centuries.

Even until fairly recent times the custom of dating from the years of the reigns of successive sovereigns was a general practice throughout Europe.

Most peoples, however, sooner or later have realized the necessity of having a fixed starting point from which all their chronological records could be counted; but the indications are that the ancient Maya were the first of all peoples to reach this important and, chronologically speaking, basic concept.

The specific events, selected by different peoples of the world at different times for the starting points of their respective chronologies, may be classified into two general groups: (1) those starting from specific historical events, and (2) those starting from obviously hypothetical, that is to say assumed, events.

The most familiar chronology of the first group is our own Christian Era, the starting point of which is the birth of Christ, our years being reckoned as B.C. (Before Christ) or A.D. (Anno Domini – 'In the Year of Our Lord'), according as they precede or follow this event.

The Greeks reckoned time by four-year periods called Olympiads from the earliest Olympic Festival of which the winner's name was known, namely, from the games held in 776 B.C., the winner of which was a certain Coroebus. The Romans took as their starting point the supposed foundation of Rome by Romulus and Remus in 753 B.C., counting their years from this event.

The Babylonian chronological epoch was called the 'Era of Nabonassar', and dated from the beginning of that king's reign in 747 B.C. The death of Alexander the Great in 325 B.C. ushered in the 'Era of Alexander'. With the occupation of Babylon in 311 B.C. by Seleucus Nicator began the so-called

'Era of the Seleucidae'. The conquest of Spain by Augustus Caesar in 38 B.C. marked the beginning of a chronology which endured for more than fourteen centuries in the Iberian Peninsula. The Mohammedans selected as the starting point of their chronology the Flight of the Prophet Mohammed from Mecca in A.D. 622, events being described in Mohammedan chronology as having occurred so many years after the Hegira or Flight. Indeed, this last chronology has persisted in Turkey down to within the past two decades, only having been partially abolished by Kemal Pasha.

It will be noted that every one of the foregoing chronological systems has for its starting point some actual historical event, the occurrence, if not the date, of which is indubitable. There are, however, other chronologies, those belonging to the second group mentioned above, that begin with an event the very nature of which renders the date of its starting point necessarily hypothetical. Here should be included those chronologies which reckon time from a supposititious date of the creation of the world.

For example, the 'Era of Constantinople', the chronological system used in the Greek Church, commences with the Creation, which is reckoned as having taken place in 5509 B.C. The Jews consider the same event as having taken place in 3761 B.C. and begin their era at that time. Perhaps the most familiar example of a chronology belonging to this second group is that of the old family Bible, which fixes the creation of man, that is the birth of Adam, as having occurred in 4004 B.C. Archbishop Usher is responsible for this chronology, having naïvely fixed the date of Adam's birth by adding the years of the generations beginning with Adam and ending with Joseph, the father of Christ as given in the Bible (the Book of Genesis and the Gospel according to St Matthew), reaching a total of 4,004 years.

While we do not know the nature of the event with which the ancient Maya began their chronology, it is practically certain that it was hypothetical rather than historical.

This is true because the zero date of the Maya chronological era, '4 Ahau 8 Cumhu', precedes their earliest contemporary records . . . i.e. for more than 3,400 years after the beginning of

their time count – that is, during the first 8 *baktuns* and nearly through the ninth *baktun* – there is not a single contemporary date; of the Maya Era the first record occurs in 8.14.3.1.12. This can mean only one thing, namely, that the starting point of Maya chronology was selected as such a very long time (probably nearly 3,000 years) *after* it was actually current time; and further, that this long blank period, devoid of contemporaneous records of any kind, probably is to be interpreted simply as a priestly approximation of past time as a whole, rather than regarded as a chronological era that had been in use for more than three thousand years before its first records appear.

It was suggested earlier that the actual inauguration of the Maya calendar probably took place in 7.0.0.0.0, or 7.6.0.0.0 of the Maya Era – either 2,760 or 2,878 years after its zero date. In view of this fact and in view of the absence of all contemporary records dating from the first 3,400 years of the Maya Era, we are forced to conclude that the astronomer-priests who devised Maya chronology selected for its starting point a date 7 *baktuns* (2,760 years) earlier than the date of its actual inauguration. Therefore it seems much more likely that Maya chronology began with some hypothetical event, rather than with an actual historical occurrence. Possibly it may have commenced with a suppositious event like the creation of the world, from which the chronologies used in the Greek and Jewish Churches, as well as in the old family Bible, are reckoned. Perhaps it may even have been counted from the supposed date of the birth of their gods, in which indefinite and remote past we must leave this question as yet undetermined.

THE INITIAL SERIES, OR LONG COUNT

The Initial Series method of dating was first so named by A. P. Maudslay, the English archaeologist and explorer. When it occurs, this time count is usually found at the beginning of the inscription; hence the name 'Initial Series', which Maudslay gave to it.

Ernst Forstemann, the German archaeologist, first worked out the details of the Initial Series count as they are presented in the codices, in 1887. J. T. Goodman, the American archaeologist,

though better known as the man who gave Mark Twain his first literary job as cub reporter on the *Virginia City Enterprise* of Virginia City, Nevada, first deciphered this time count in the inscriptions on the monuments in 1890, basing his work on Maudslay's magnificent reproduction of Maya sculptures. Goodman's discoveries, made quite without knowledge of Forstemann's slightly earlier investigations, are in perfect agreement with the latter's findings and prove beyond all possibility of doubt that the ancient Maya Initial Series time count is now as clearly understood as our own Gregorian calendar.

A brief description of a typical Initial Series follows: At the beginning stands a large glyph, usually four times as large as the other glyphs in the inscription, which has been called the introducing-glyph because it 'introduces' the Initial Series time count (Figure 5.11). The only part of this large initial glyph which varies in the different inscriptions (except for minor differences in style) is the central element, of which there are nineteen different forms, one for each of the nineteen divisions or months of the Maya calendar year, already described. These variable central elements, in all probability, as suggested in the preceding chapter, are the name-glyphs of the deities who presided over the nineteen divisions or months of the year. The form recorded in any given introducing-glyph is probably that of the name-glyph of the specific deity who was patron of the month in which the corresponding Initial Series terminal date fell.

Maya inscriptions are to be read from left to right and top to bottom in pairs of columns so that the two columns of smaller glyphs immediately below the large introducing-glyph in Figure 5.11 are to be read from left to right and top to bottom.

The first five glyphs in these two columns, following this order of reading, are the number of baktuns, katuns, tuns, uinals and kins involved in writing this particular date, that is, the periods of 144,000, 7,200, 360, 20 and 1 days, respectively, that have elapsed from the starting point of Maya chronology – the date 4 Ahau 8 Cumhu, described above but not actually recorded – down to the date reached by this particular Initial Series number.

Although the unit of the Maya Initial Series is the day, while

that of our own Christian chronology is the year, the two systems are not unlike in their respective methods of record. For example, when we write the date Monday, 31 December, A.D. 1945, we mean that 1 period of one thousand years, 9 periods of one hundred years, 4 periods of ten years and 5 periods of one year have elapsed since the birth of Christ, the starting point of our own chronology, called 'Anno Domini', (abbreviated to A.D.) meaning 'In the Year of Our Lord', to reach a day Monday, which was the 31st day of the month of December.

Similarly, when the ancient Maya wrote the Initial Series 9.17.0.0.0 13 Ahau 18 Cumhu, shown in Figure 5.11, they meant that 9 periods of 144,000 days (9 baktuns), 17 periods of 7,200 days (17 katuns), 0 periods of 360 days (0 tuns), 0 periods of 20 days (0 uinals) and 0 periods of 1 day (0 kins) had elapsed since the starting point of their chronology, 4 Ahau 8 Cumhu (unexpressed, but assumed), until the day '13 Ahau' which occupied the month-position '18 Cumhu', was reached.

The day of this terminal date – here '13 Ahau' – is usually found (nine out of ten times) in the sixth position after the introducing-glyph, or immediately following the fifth and last time period (the kins) of the Initial Series number (Figure 5.11).

In almost all cases, the glyph immediately following the day of the Initial Series terminal date, the one in the seventh position after the large introducing-glyph (Figure 5.11), is called Glyph G, and has already been described in the preceding chapter. It has nine forms, one corresponding to each of the *Bolontiku*, or 'Nine Gods of the Lower World', referring in each inscription to that particular one of these nine gods who was the patron of the specific day of the Maya Chronological Era, reached by the accompanying Initial Series number. In the Initial Series here described, the day-patron was the sun-god, patron of the Ninth Day (Figure 5.11). And following this last glyph is another of unknown meaning, Glyph F, the last sign of the Initial Series, except for the corresponding month part which follows immediately after the last glyph of the Supplementary Series.

Initial Series Introducing-Glyph

Grotesque head in centre is the only variable element of this sign. This is the name-glyph of the deity who is patron of the month (here Cumhu) in which the Initial Series terminal date falls

9 baktuns (9 × 144,000 days = 1,296,000 days)	17 katuns (17 × 7,200 days = 122,400 days)
0 tuns (0 × 360 days = 0 days)	0 uinals (0 × 20 days = 0 days)
0 kins (0 × 1 day = 0 days)	13 Ahau (day reached by counting forward above total of days from starting point of Maya Era)
Glyph G9 Name-glyph of the deity who is patron of the Ninth Day in the nine-day series (The Nine Gods of Lower World)	Glyph F Meaning unknown
Glyphs E and D Glyphs denoting the moon age of the Initial Series terminal date, here 'new moon'	Glyph C Glyph denoting position of current lunar month in lunar half-year period, here the 2nd position
Glyph X3 Meaning unknown	Glyph B Meaning unknown
Glyph A9 Current lunar month, here 29 days in length. Last glyph of the Supplementary Series	18 Cumhu (month reached by counting forward above total of days from starting point of Maya Era). Last glyph of the Initial Series

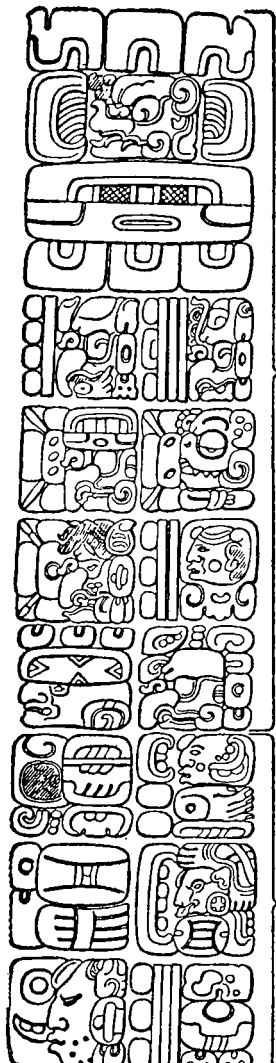


Fig. 5.11. Examples of an Initial and a Supplementary Series: east side of Stela E, Quirigua, Department of Izabal, Guatemala.

THE SUPPLEMENTARY SERIES, OR MOON COUNT

Between Glyph F and the corresponding month part (the position of the day in the calendar year), here '18 Cumhu', of the Initial Series, there usually stands a group of six glyphs which have been called the Supplementary Series, since they supplement the meaning of the Initial Series. They give information (1) about the moon on the date recorded by the accompanying Initial Series, that is the age of the moon, probably counted from the new moon; (2) the length of the particular lunar month in which the Initia Series date fell, that is, whether it was composed of 29 or 30 days, here 29 days; (3) the number of the particular lunation in the lunar half-year period, here 2; and a few other as yet undetermined points. Finally, following the last glyph of the Supplementary Series is the month-glyph of the Initial Series terminal date, in the text illustrated here, '18 Cumhu' (Figure 5.11).

And thus by means of their simple, yet highly efficient, vigesimal numerical system, as expressed in their amazingly accurate Initial Series or Long Count, the ancient Maya were able to fix any given date in their chronology with such a high degree of precision that it could not recur, fulfilling all the prescribed conditions, until after a lapse of 374,440 years – a truly colossal achievement for any chronological system, whether ancient or modern.

6

THE PERUVIAN QUIPU

(Highest development in the fifteenth and sixteenth centuries)

THE Peruvian quipu or knot record was a remarkable development and extension of the primitive practice of recording numbers and reckoning by means of knots in cord. Tradition assigns the origin of the quipu to very early times. Authentic examples of the ancient quipus have been found principally in the coastal desert plains of Peru between the Andes Mountains and the Pacific Ocean, in the vicinity of the City of Lima. The discovery of knot records in Peru in the sixteenth century was all the more astonishing inasmuch as the quipus were found in common use at a time when the centralized governmental organization was highly complex. The Peruvian people, despite their possession of a rich and expressive language, had no system of writing. Knowledge of the quipu was restricted to the Inca tribesmen who held dominion over a widespread area in the north-eastern part of South America, and to their selected functionaries. Chosen on the basis of their capabilities, the latter were trained in the uses of the quipu and were paid, usually, through exemption from the payment of taxes.

When the Spaniards first observed the use of the quipu among the Incas in the sixteenth century, they found the system of recording numbers by knots with its decimal place values in a highly developed form. The thickness of the strands and their colour were given special meanings. White represented silver, yellow represented gold and red, soldiers or war. Each district under the rule of the Incas had its own specially trained *Quipu-camayu* (he who has charge of the accounts). Some villages had as many as twenty or thirty such individuals whose duty it was to record by means of the quipu amounts of tribute, the size and type of flocks, items relating to the administration of justice and so on. A network of communication posts to which runners (*chasquis*) were assigned facilitated the flow of information to the capital at Cuzco from all parts of the Inca dominion. Frequently the report which was coded n the quipu was supplemented by carefully memorized oral messages. Although Peruvian culture reached a level of considerable complexity, involving a strict class system, governmental records of crops and herds,

and amazing feats of architecture, no writing materials or written symbols had been evolved. Hence, the quipus fulfilled a double purpose, serving on the one hand for mathematical accounting, and on the other as memoranda of historical events, ceremonies, negotiations for peace or war, and the like.

Recently, archaeological researches have revealed great new finds. Objects of great artistry made of gold, silver, copper and *tumbaga*, an alloy of these metals, have been unearthed. Great mountain fortresses, reached by perilous stone steps and guarded by stone walls on steep slopes, have been revealed with their artificial terraces, hanging gardens, irrigation canals and stone sculptures. The metal work of the ancient Peruvian goldsmith, as well as the sculptured statues, express the worship of deities requiring human sacrifice, though it is known that this practice in later times was gradually being abandoned. One of the most important of the early historical accounts of the Inca Civilization is the *Royal Commentaries* written by Garcilasso de la Vega (b. 1539) whose father was a Spanish cavalier and whose mother was an Inca princess. Garcilasso lived among the Incas until he was twenty years of age and his intimate knowledge of the Incas is one of our most important sources of information concerning the Inca Civilization. His work describes the organization of the Inca state, the selection of officials forming the liaison between the Incas and the general population, and the Indian way of life in which money was unknown and barter was the means of trading. He laid great stress on the gentle nature of the people, their honesty and fair dealing.

In the centuries succeeding the Spanish conquest, the spread of the knowledge of writing generally eliminated the need for the quipu. Nevertheless, its use persisted in some areas and examples of the quipu in its modern form can still be found.

THE ANCIENT QUIPU

From The Ancient Quipu or Peruvian Knot Record
by L. Leland Locke

DESCRIPTION OF THE QUIPU

The characteristics of the ancient quipu, both as described by the Spanish writers and exemplified in extant specimens, are:

1. A main cord varying in length from a few centimetres to a metre or more.

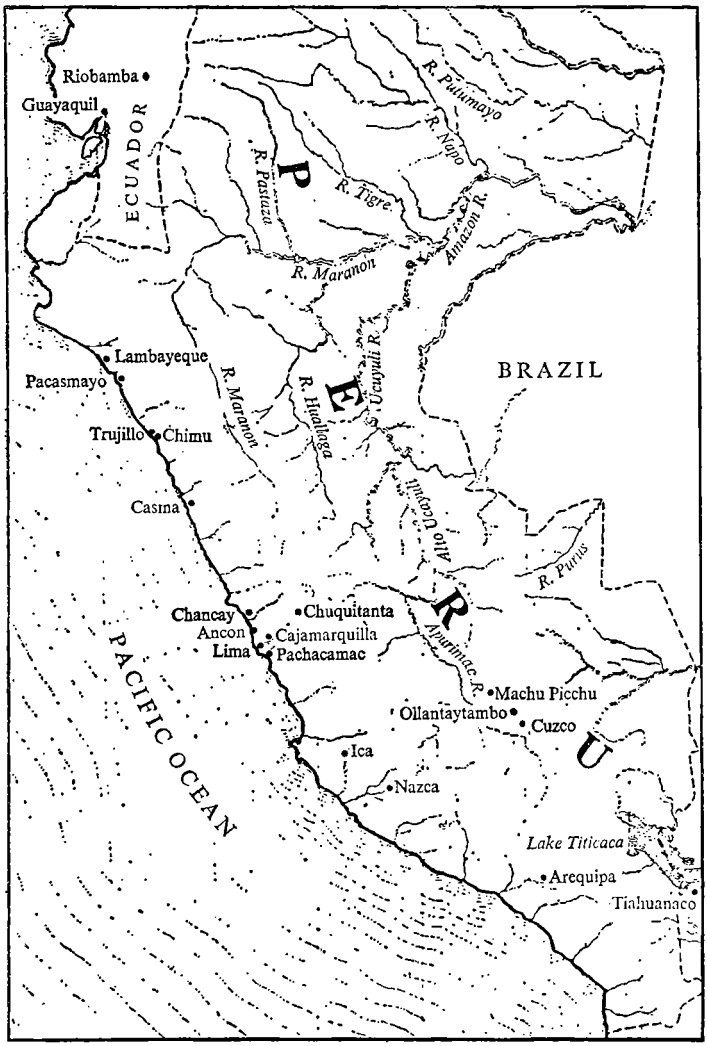


Fig. 6.1. Map showing important archaeological sites in Peru.

2. Attached to the main cord are pendent cords seldom exceeding 0·5 metre in length. These vary in number from one or two, to a hundred or more, in existing specimens. The manner of forming the cords is to spin a cord of twice the length desired, then double it, with a loop at one end, as seen in Figure 6.2, and twist the two strands together. Usually a knot is tied in the free end of the cord, and in most cases the cord tapers off at the end, indicating the running out of the material in spinning. Both cotton and wool were used, natural white and buff being available in both materials.

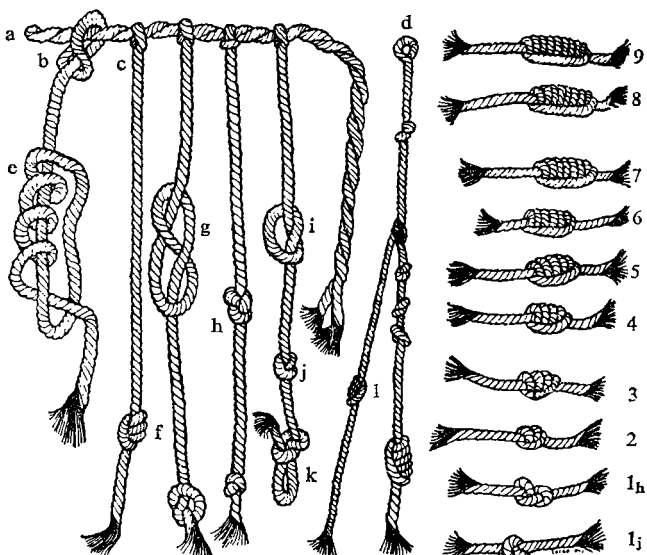


Fig. 6.2. Method of tying knots.

The pendent strands are attached to the main cord by passing the free end over the main cord and through the loop formed by doubling, and then drawing taut.... The pendent strands are variously distributed along the main cord, usually as close together as possible, sometimes with small spaces between the groups, or the single strands.

3. Knots are tied in the pendent strands at varying distances from the main cord. In the better types these groups of knots are arranged roughly in rows across the quipu.

4. In nearly all of the ancient quipu short subsidiary cords are attached to the pendent strands, upon which are indicated numbers that disturb the main count of the quipu.

The subsidiary cord is attached to the pendent strand in the same manner as the pendent strand is fastened to the main cord, except that in most cases it is inserted between the two strands and therefore looped around half of the pendent strand. This is illustrated in Figure 6.2*d*.

5. Character of the knots. Figure 6.2 shows the forms of the knot and the mode of tying which exist in the specimens studied. The single or overhand knot (*i*, *j*), indicates 1 if it is in the row farthest from the main cord, 10 if it occurs in the next nearer row, 100 in the next row, etc. Not more than nine single knots are found in one group, the number system being strictly decimal. The figure eight knot occurs frequently.

The long knot, used to express the repetition of units of the same order in place of a cluster of single knots, has been likened by Mr Frank H. Cushing to the appearance of the closed fist. It is formed by tying the overhand knot and passing the end through the loop of the knot as many times as there are units to be denoted. One end is then drawn taut, thus coiling the other about the strand the required number of times. There seems to have been no fixed practice as to which end is drawn taut, the upper fixed end, or the lower free end. As this would lower or raise the knot on the cord, it is possible that the device was used to keep the knots of one order relatively the same distance from the main cord. In some of the cruder specimens it is apparent that both ends have been pulled, as described earlier. The loop has apparently no numerical significance, but, from the manner of its appearance, it may have had some such use as the red line used by book-keepers in closing an account.

ANALYSIS OF QUIPU NO. 1

This specimen (Figure 6.3) in the Bandelier collection in the American Museum of Natural History is one of three from

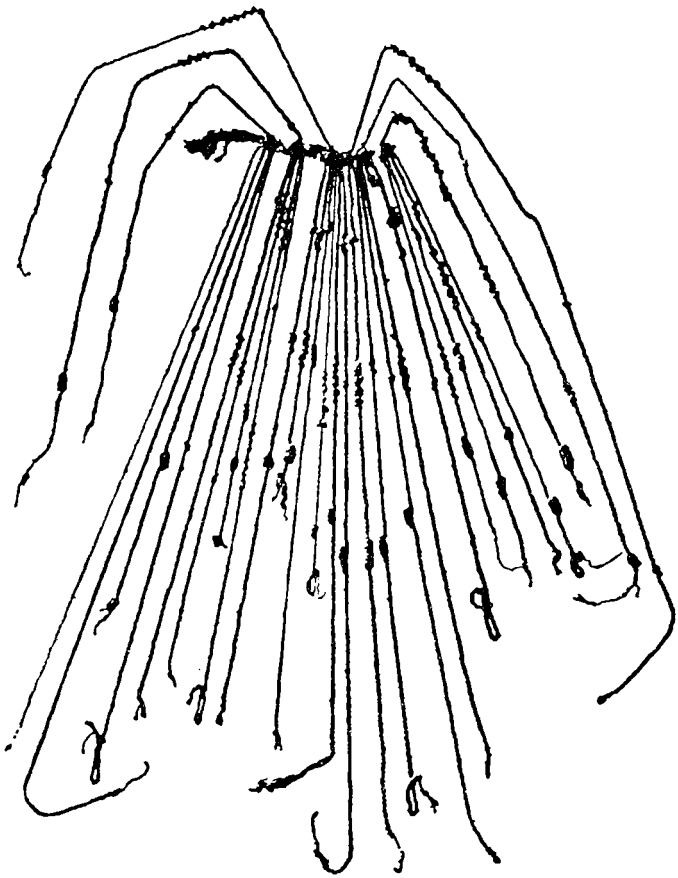


Fig. 6.3. Example of the highest development of the quipu. Quipu No. 1, from Chancay.

Huando, north of Lima. This village was of the Chancay tribe, which was subdued by the Inca and from which tribute was levied. This is an example of the most highly developed form of the quipu, and is important in that it gives an accurate key to the numerical character of the knots.

The characteristics of this and the other two specimens from this locality are:

1. The use of the long knot for ones only, and groups of single knots for the higher orders.
2. The use of the loop on the first (or last) cord of a group.
3. The pendent cords are grouped (in this specimen in fours, in the others in sixes) by passing a top cord through the top loops of the four (or six) pendent strands. See Figure 6.4.

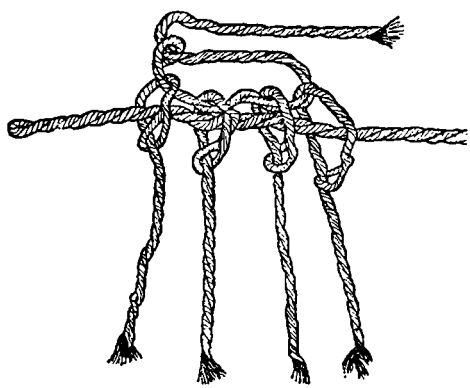


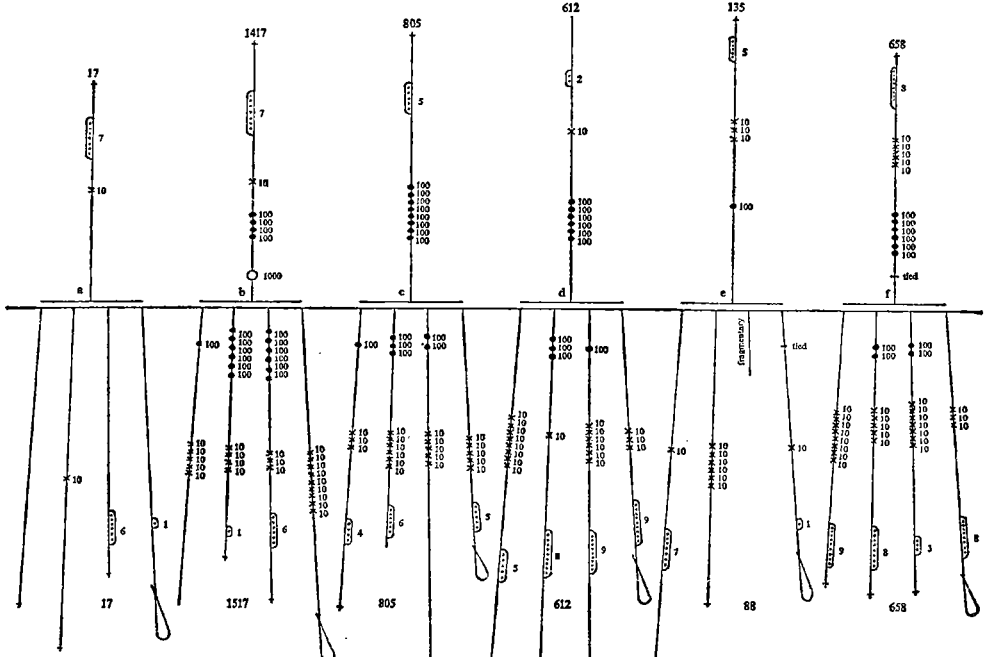
Fig. 6.4. Mode of attaching the top cord upon which the numbers of the pendent strands are summed.

4. *The top strand sums the numbers on the group of pendent strands through which it is passed.*
5. The ones, tens, and hundreds are arranged in rows across the quipu, 'as a good accountant arranges them in columns'.
6. The auxiliary strand appears in two of the three specimens of this type.

A comparison with the Uhle or Bastian modern quipus would suggest that this specimen is a record of four types of things covering six years or periods.

Figure 6.3 shows this quipu and Figure 6.5 is a line diagram of the arrangement of knots.

The accompanying table gives the knot readings and colours.



The pendent strands are grouped in fours, each group being tied with a top strand. The top strand SUMS the numbers on the four pendent strands.
_____ indicates Ones, each dot is a single one. Each X indicates a ten. Each ● indicates a hundred. O indicates a thousand.

Table of knot readings and colours, Quipu No. 1
(Compare with Figures 6.3 and 6.5)

Strands	1000's	100's	10's	1's	Colour
Main cord	—	—	—	—	White and dark brown
1	—	—	—	—	Brown
2	—	—	1	—	White
3	—	—	—	6	White
4	—	—	—	1	White
Sum	—	—	1	7	
a	—	—	1	7	Very light brown
1	—	1	5	—	White and reddish brown
2	—	6	4	1	White and light brown
3	—	6	3	6	White and dark brown
4[1]	—	—	9	—	White and dark brown
Sum	1	13	21	7	
	—	5	1	7	
b	1	4	1	7	White and light brown
1	—	1	3	4	Brown
2	—	3	6	6	Brown
3	—	2	5	—	Brown
4	—	—	5	5	Brown (darker)
Sum	—	6	19	15	
	—	8	0	5	
c	—	8	—	5	Light brown, reddish

1. Fragmentary subsidiary cord (?); discrepancy of 100 in count.

Table of knot readings and colours, Quipu No. 1 (continued)

Strands	1000's	100's	10's	1's	Colour
1	—	—	8	7	Blue
2	—	3	1	9	Blue
3	—	1	6	9	Blue
4	—	—	3	7	Blue
Sum	—	4	18	32	
	—	6	1	2	
d	—	6	1	2	Light brown, yellowish
1	—	—	1	7	Light blue and white
2	—	—	6	—	Light blue and white
3	Frag.	—	4(?)	7(?)	Light blue and white
4	—	—	1	1	Light blue and white
Sum	—	—	12(?)	15(?)	
	—	1	3	5	
e	—	1	3	5	Light and dark brown
1	—	—	8	9	Light brown
2	—	2	5	8	Light brown
3	—	2	7	3	Light brown
4	—	—	3	8	Light brown
Sum	—	4	23	28	
	—	6	5	8	
f	—	6	5	8	Light brown

MODERN QUIPU

Two specimens of the modern quipu have been secured by Dr Max Uhle. One of these, which is on exhibition at the University

of Pennsylvania Museum, was described by Dr Uhle in the Bulletin of the Free Museum of Science and Art of the University of Pennsylvania, December, 1897. (See Figure 6.6.)

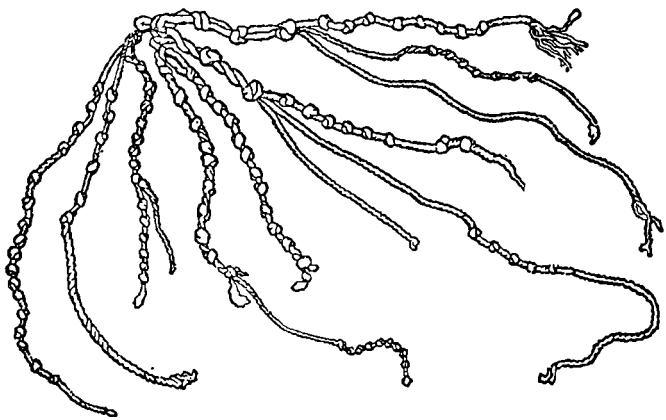


Fig. 6.6. Drawing of a modern quipu. After Uhle.

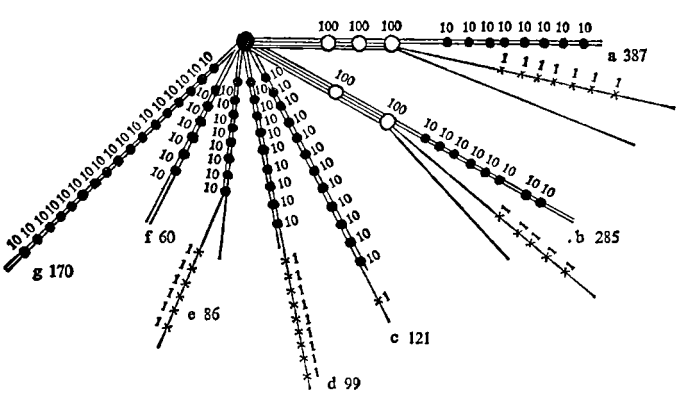


Fig. 6.7. Diagram of Quipu No. 40, shown in Figure 6.6.

The quipu was obtained in April 1895 from an Indian, who was in charge of the flocks of a farm at Cutusuma, near Lake Titicaca, and consists of white strings of sheep wool. The Indian informed Dr Uhle that the number of females is indicated on the border of the quipu, and that of the males in the centre.

The number recorded on this quipu (as is seen in the key drawing and key) is 387 sheep, 285 rams, 121 lambs of the second set of the year (the first set having been counted in with the adults of the year), 99 lambs of the third set, 86 and 60 and 170 milking sheep of the three sets of lambs.

The illustration shows that the decimal system is used, the knots indicating hundreds, tens and ones, but that the cord is divided to record the tens on a string finer than that used for the hundreds, and still further divided to record the ones. Further the knots are all single knots. The subdivision of the cords is not found on any specimen of the ancient quipu which has been examined by the writer.

The other specimen secured by Dr Uhle was forwarded to Professor Bastian, at Berlin, and was described by him from notes furnished by Dr Uhle (*Ethnologisches Notizblatt*, 1895, 2, 80, Plate III). This quipu is from Challa, Island of Titicaca, and was secured in the same manner as the other, and is a three-part quipu. The first, part I, is a register of the flock in the care of the shepherd. The second and third parts, each of three strands, represent on the first part the males, and on the second part the ewes which the shepherd had received in three successive months from another shepherd. The record for each month is on a separate strand. The construction of this quipu is similar to that from Cutusuma in the use of the single knot and subdivision of the string. The maker utilizes colours, which is not done in the Cutusuma quipu, and also in two parts of the main quipu the subsidiary strand is used to indicate an exception which disturbs the record. (Strands I*b* and II represent, respectively, the number of sheep eaten by the shepherd and by the owner of the farm.) This was a familiar feature of the ancient quipu. (Cf. Figures 6.8, 6.9, 6.10.)

A comparison of these modern quipu with the ancient reveals the following facts:

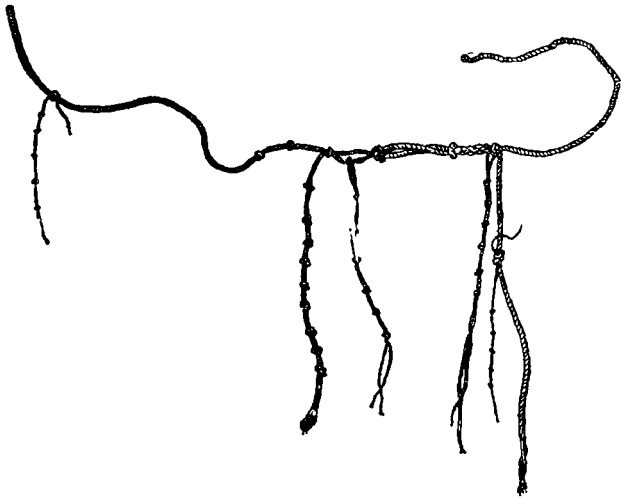


Fig. 6.8. Section 1 of a modern quipu (Bastian).

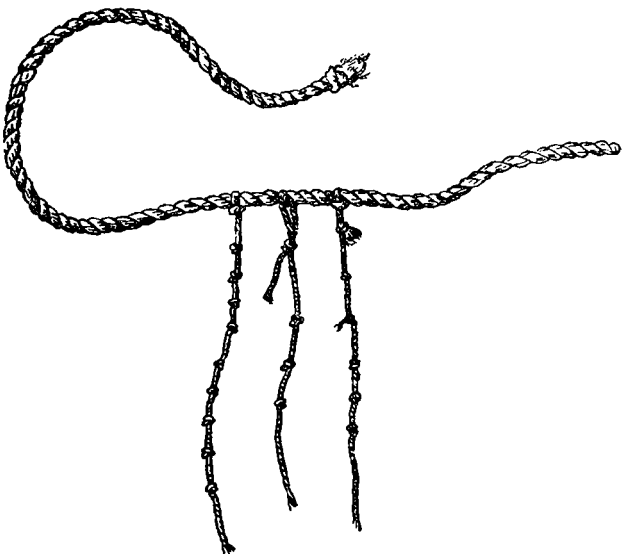


Fig. 6.9. Section 2 of a modern quipu (Bastian).

1. The decimal system and the relative placing of the hundreds, tens and ones is preserved.

2. The subsidiary strand, a unique property of the ancient quipu, is preserved.

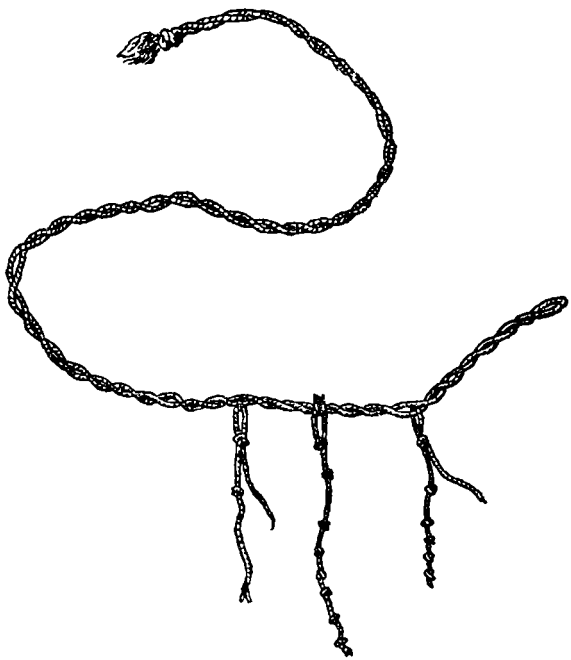


Fig. 6.10. Section 3 of a modern quipu (Bastian).

3. The tying of the hundred on a double cord of two double strands, the separation of the original and the tying of the tens on one of the double strands and the still further separation into single strands, upon one of which the ones are tied, is not found in any of the ancient specimens examined. Dr Brehm gives this as a property of the ancient quipu. This may be based on conjecture, or on statements of contemporary writers, or on the examination of specimens. The writer has been unable to find any such corroborative evidence.

4. The main strand is found in parts II and III of the Bastian quipu but is not used in either the principal part I, nor in the Uhle quipu.

5. The long knot and the mode of attaching to the main strand have been lost.

EXCERPTS

From the Royal Commentaries *by Garcilasso de la Vega*

The ordinary judges gave a monthly account of the sentences they had pronounced to their superiors, and these to others, there being several grades of judges, according to the importance of the cases.

The way of making these reports to the Ynca, or to those of his Supreme Council, was by means of knots, made on cords of various colours, by which means the signification was made out, as by letters. The knots of such and such colours denoted that such and such crimes had been punished, and small threads of various colours attached to the thicker cords signified the punishment that had been inflicted, and in this way they supplied the want of letters.

*

The accountants were very accurate. Quipu means to knot or a knot, and it was also understood as an account, because the knots supplied an account of everything. The Indians made strings of various colours. Some were all of one colour, others of two combined, others of three, others more; and these colours, whether single or combined, all had a meaning. The strings were closely laid up in three or four strands, about the girth of an iron spindle, and three-quarters of a vara (about two feet) long. They were strung on a thicker cord from which they hung in the manner of a fringe. The thing to which a string referred was understood by its colour, for instance a yellow string referred to gold, a white to silver, and a red one to soldiers. Things which had no colour were arranged according to their importance, beginning with that of most consequence, and proceeding in order to the most insignificant; each under its generic head, such as the different kinds of grain under corn, and the pulses in the same

way. We will place the cereals and pulses of Spain in their order, as an example. First would come wheat, next barley, next beans, next millet. In the same way, when they recorded the quantity of arms. First they placed those that were considered the most noble, such as lances, next darts, next bows and arrows, then shields, then aces and then slings. In enumerating the vassals, they first gave account of the natives of each village, and next of those of the whole province combined. On the first string they put only men of sixty and upwards, in the second those of fifty, in the third those of forty and so on down to babies at the breast. The women were counted in the same order.

Some of these strings had other finer ones of the same colour attached to them, to serve as supplements or exceptions to the chief record. Thus if the main strand of men of a certain age had reference to married people, the supplementary strand gave the number of widowers of the same age in that year. For these accounts were made up annually and only related to one year. The knots indicated units, tens, hundreds, thousands and tens of thousands, but they rarely or never went beyond that; because each village was taken by itself, and each district, and neither ever reached to a number beyond tens of thousands, though there were plenty within that limit. But if it were necessary to record a number equal to hundreds of thousands, they could do it, for in their language they were able to express any number known in arithmetic; but as they had no occasion to go beyond tens of thousands, they did not use higher numbers. These numbers were counted by knots made on the threads, each number being divided from the next. But the knots for each number were made together in one company, like the knots represented in the girdle of the ever blessed Patriarch, St Francis; and this could easily be done as there were never more than nine, seeing that the units, tens, etc., do not exceed that number. On the uppermost knot they put the highest number, which was tens of thousands, and so on to the units. The knots of each number, and each thread, were placed in a line with each other exactly in the way a good accountant places his figures to make a long addition sum.

These knots or quipus were in charge of Indians, who were

called *Quipu-camayu,* which means 'He who has charge of the accounts'. Although there was, at the time, little difference of character among the Indians, because owing to their gentle dispositions and excellent government all might be called good, yet the best and those who had given the longest proofs of their fitness were selected for these and other offices. They were not given away from motives of favouritism, because these Indians were never influenced by such considerations, but from considerations of special fitness. Nor were these either sold nor farmed out, for they knew nothing of renting, buying or selling, having no money they exchanged one article of food for another, and no more; for they neither sold clothes, nor houses nor estates.

The *Quipu-camayus* being so trustworthy and honest, as we have described – their number was regulated according to the population in each village; for however small the village might be, there were four accountants in it, and from that number up to twenty or thirty; though all used the same register. Thus as only one account was kept, one accountant would have been sufficient; but the Incas desired that there should be several in each village to act as checks upon each other, and they said that where there were so many all must be in fault or none.

MATHEMATICS FROM THE EAST

7

WAN WANG

(Twelfth century B.C.)

WAN WANG (King Wan), Chinese Prince, viceroy of the Chou province in the twelfth century B.C., admired as a model of all that is good, beloved for his wisdom and restraint, is considered the founder of the Chou dynasty, even though he himself never held the throne. Wrongly suspected of disloyalty by Chou Hsin, the last ruler of the Shang dynasty, he was imprisoned by Hsin in 1143 B.C. and in immediate danger of execution. Wan was rescued by a party of his followers led by his son, Wu Wang. Soon afterwards, Wu Wang and a coalition of princes drew up a bill of particulars listing their charges of misrule against Hsin. Numerous specific instances of Hsin's unspeakable cruelty were given. In the ensuing war undertaken to overthrow the Shang ruler, the revolt against the Shang succeeded. With the assistance and support of his distinguished brother Chou Kung, the Duke of Chou (Kau), Wu Wang was established as the first emperor of the Chou dynasty.

Wan Wang had been imprisoned for almost a year. During that time he wrote the textual part of the *Yi King*, a book which came to be regarded by the Chinese with great reverence. In the *Yi King* Wang combined the eight *kwa* or trigrams of *Fu-hsi* by twos into sixty-four hexagrams, adding to each his statement of the meaning of the figure. His son, the Duke of Chou (Kau), is credited with the explanation of each of the lineal components of each hexagram. Ten *Appendixes* added to the writings of Wan Wang and his son form the latter part of the *Yi King*. The *Appendixes* are attributed to Confucius (*c.* 550–479 B.C.). Wan Wang and his son, both skilled in divination, intended that their writings be used for divining, and the meanings set forth were deliberately made obscure to avoid abuse by unworthy individuals of the power to be derived from a true knowledge. The *Yi King*, also called the *Book of Changes*, escaped destruction in the general book-burning ordered by the despot of the Ch'in at the end of the third century B.C., as books of divination were expressly exempted from the burning. The interpreta-

tion of the text given by Confucius in the *Appendixes* was philosophical. Great pride and respect is felt by the Chinese for the *Yi King* as a fundamental book of their philosophy. A long list of scholars gained distinction as expounders of the meaning of the *Yi King* and its *Appendixes*. The imperial edition of the *Yi King* published in 1718 contained extracts from commentaries on it by more than two hundred writers dating from the second century B.C.

It is stated by some that Wan Wang, the Duke of Chou and Confucius were all editors, and that Fu-hsi was the original author. Fu-hsi (*c.* 3020 B.C.), who belonged to the Mythical Period of Chinese history, before the invention of the plough, when men lived mainly by hunting and fishing, is credited with the invention of the practice of divination. The diagrams, contemporary with the invention of writing, became the expression of a philosophy. The first principle to be expressed in the *Yi King* was the philosophy of the dual, involving the strong element and the weak element in nature. As time went on this was contracted into a single antithesis and then expanded into a system containing a number of elements. Rudiments of ancient Chinese mathematics are found in the *Yi King*.

THE YI KING

From The Sacred Books of China *translated by James Legge*

INTRODUCTION

In (the system of) the Yi there is the Great Extreme, which produced the two I (Elementary Forms). These two Forms produced the four Hsiang (Emblematic Symbols); which again produced the eight Kwa (or Trigrams). The eight Kwa served to determine the good and evil (issues of events), and from this determination there ensued the (prosecution of the) great business of life.

The two elementary Forms, the four emblematic Symbols, and the eight Trigrams can all be exhibited with what may be deemed certainty. A whole line (———) and a divided (—— ——) were the two I. These two lines placed over themselves, and each of them over the other, formed the four Hsiang: ═══; ═ ═; ═ ═; ═ ═. The same two lines placed successively over these Hsiang, formed the eight Kwa, exhibited above.

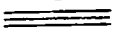
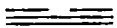
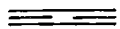
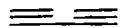
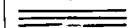
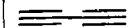
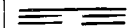
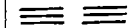
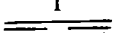
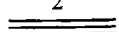
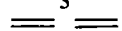
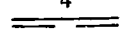
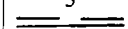
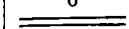
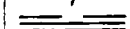
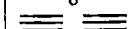
1	2	3	4	5	6	7	8
							
khien	tui	lî	*kăn*	sun	khân	kăn	khwăn
Heaven, the sky.	Water, collected as in a marsh or lake.	Fire, as in lightning; the sun.	Thunder.	The wind; wood.	Water, as in rain, clouds, springs, streams, and defiles. The moon.	Hills, or mountains.	The earth.
S.	S.E.	E.	N.E.	S.W.	W.	N.W.	N.
Untiring strength; power.	Pleasure; complacent satisfaction.	Brightness; elegance.	Moving, exciting power.	Flexibility; penetration.	Peril; difficulty.	Resting; the act of arresting.	Capaciousness; submission.

Fig. 7.1. Fu-hsi's trigrams.

1	2	3	4	5	6	7	8
							
lî	sun	*kăn*	kăn	khân	*khien*	tui	khwăn
Second daughter.	Oldest daughter.	Oldest son.	Youngest son.	Second son.	Father.	Youngest daughter.	Mother.
S.	S.E.	E.	N.E.	N.	N.W.	W.	S.W.

A strong and a weak line were manipulated together (till there were the 8 trigrams,) and those 8 trigrams were added each to itself and to all the others (till the 64 hexagrams were formed).

Fig. 7.2. King Wan's trigrams.

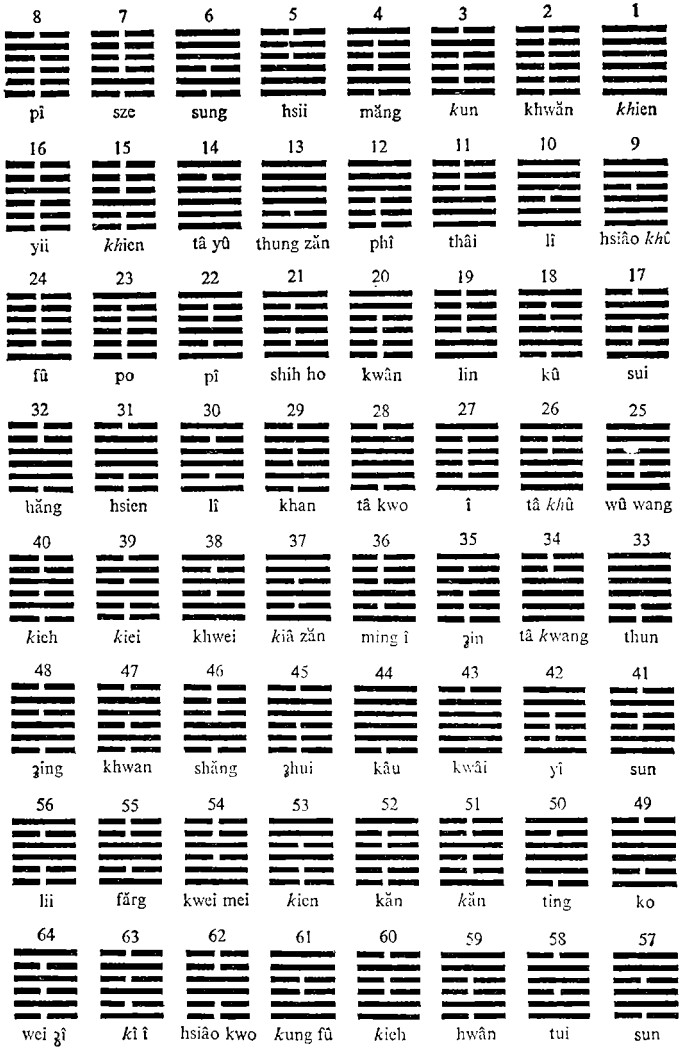

8 7 6 5 4 3 2 1
pî sze sung hsii măng kun khwăn khien
16 15 14 13 12 11 10 9
yii khien tâ yû thung zăn phî thâi lî hsiâo khû
24 23 22 21 20 19 18 17
fû po pî shih ho kwân lin kû sui
32 31 30 29 28 27 26 25
hăng hsien lî khan tâ kwo î tâ khû wû wang
40 39 38 37 36 35 34 33
kich kiei khwei kiâ zăn ming î ʒin tâ kwang thun
48 47 46 45 44 43 42 41
ʒing khwan shăng ʒhui kâu kwâi yî sun
56 55 54 53 52 51 50 49
lii fărg kwei mei kien kăn kăn ting ko
64 63 62 61 60 59 58 57
wei ʒî kî î hsiâo kwo kung fû kieh hwân tui sun

Fig. 7.3. The hexagrams in the order in which they appear in the Yi, and were arranged by King Wan.

Heaven produced the spirit-like things (the tortoise and the divining plant), and the sages took advantage of them. (The operations of) heaven and earth are marked by so many changes and transformations and the sages imitated them (by means of the Yi). Heaven hangs out its (brilliant) figures, from which are seen good fortune and bad, and the sages made their emblematic interpretations accordingly. The Ho gave forth the scheme or map, and the Lo gave forth the writing, of (both of) which the sages took advantage.

The words with which we have at present to do are – 'The Ho (that is, the Yellow River) gave forth the Map.' This map, according to tradition and popular belief, contained a scheme which served as a model to Fu-hsi in making his 8 trigrams. Apart from this passage in the Yi King, we know that Confucius believed in such a map, or spoke at least as if he did.[1] In the 'Record of Rites' it is said that 'the map was borne by a horse';[2] and the thing, whatever it was, is mentioned in the Shu as still preserved at court, among other curiosities, in 1079 B.C.[3] The story of it, as now current, is this, that a 'dragon-horse' issued from the Yellow River bearing on its back an arrangement of marks from which Fu-hsi got the idea of the trigrams.

The map that was preserved in the eleventh century B.C., afterwards perished, and though there was much speculation about its form from the time that the restoration of the ancient classics was undertaken in the Han dynasty, the first delineation of it given to the public was in the reign of Hui Tsung of the Sung dynasty (A.D. 1101–1125). The most approved scheme of it is shown in Figure 7.4.

It will be observed that the markings in this scheme are small circles, pretty nearly equally divided into dark and light. All of them whose numbers are odd are light circles – 1, 3, 5, 7, 9; and all of them whose numbers are even are dark – 2, 4, 6, 8, 10. This is given as the origin of the numbers of heaven and earth. The difference in the colour of the circles occasioned the distinction of them and of what they signify into Yin and Yang, the dark and the bright, the moon-like and the sun-like; for the sun is called the Great Brightness (Thai Yang), and the moon the Great Obscurity (Thai Yin). Fu-hsi in making the trigrams, and King

1. Analects IX, viii. 2. Li *Ki* VIII, iv, 16. 3. Shu V, xxii, 19.

Wan, if it was he who first multiplied them to the 64 hexagrams, found it convenient to use lines instead of the circles: the whole line (———) for the bright circle (○), and the divided line (— —) for the dark (●). The first, the third and the fifth lines in a hexagram, if they are 'correct' as it is called, should all be whole, and the second, fourth and sixth lines should all be divided. Yang lines are strong (or hard), and Yin lines are weak (or soft). The former indicate vigour and authority; the latter, feebleness and submission. It is the part of the former to command; of the latter to obey.

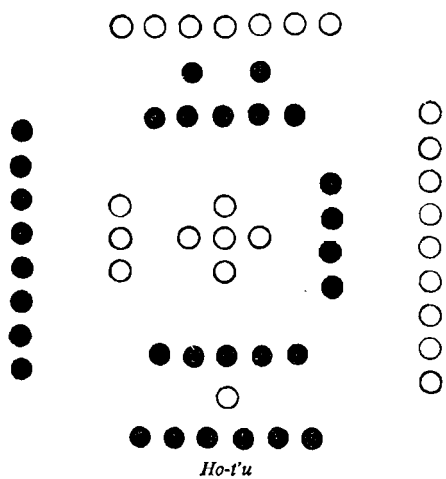
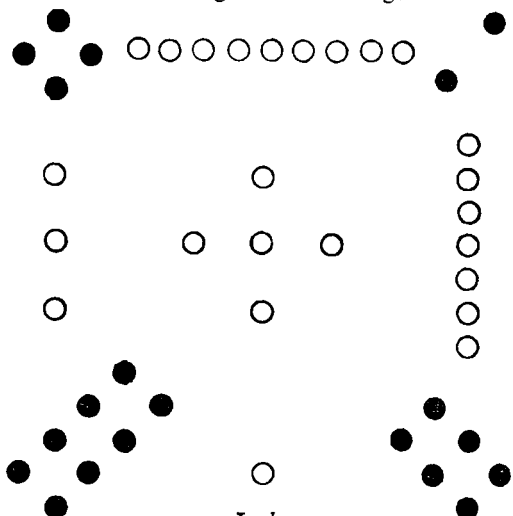


Fig. 7.4. The form of the river map.

'The Lo produced the writing.' This writing was a scheme of the same character as the Ho map, but on the back of a tortoise, which emerged from the river Lo, and showed it to the Great Yu, when he was engaged in his celebrated work of draining off the waters of the flood, as related in the Shu. To the hero sage it suggested 'the Great Plan', an interesting but mystical document of the same classic, 'a Treatise', according to Gaubil, 'of Physics, Astrology, Divination, Morals, Politics and Religion',

the great model for the government of the kingdom. The accepted representation of this writing is the following:



Lo-shu

Fig. 7.5. Lo-shu.

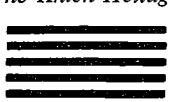
But substituting numbers for the number of marks, we have

4	9	2
3	5	7
8	1	6

in which the numbers from 1 to 9 are arranged so as to make 15 in whatever way we add them.

TEXT SECTION I

I. The Khien Hexagram



EXPLANATION OF THE ENTIRE FIGURE BY KING WAN

Khien (represents) what is great and originating, penetrating, advantageous, correct and firm.

EXPLANATION OF THE SEPARATE LINES BY THE DUKE OF KAU

1. In the first (or lowest) line, undivided, (we see its subject as) the dragon lying hid (in the deep). It is not the time for active doing.

2. In the second line, undivided, (we see its subject as) the dragon appearing in the field. It will be advantageous to meet with the great man.

3. In the third line, undivided, (we see its subject as) the superior man active and vigilant all the day, and in the evening still careful and apprehensive. (The position is) dangerous, but there will be no mistake.

4. In the fourth line, undivided, (we see its subject as the dragon looking) as if he were leaping up, but still in the deep. There will be no mistake.

5. In the fifth line, undivided, (we see its subject as) the dragon on the wing in the sky. It will be advantageous to meet with the great man.

6. In the sixth (or topmost) line, undivided, (we see its subject as) the dragon exceeding the proper limits. There will be occasion for repentance.

7. (The lines of this hexagram are all strong and undivided, as appears from) the use of the number nine. If the host of dragons (thus) appearing were to divest themselves of their heads, there would be good fortune.

*

The text under each hexagram consists of one paragraph by King Wan, explaining the figure as a whole, and of six (in the case of hexagrams 1 and 2, of seven) paragraphs by the duke of Kau, explaining the individual lines. The explanatory notices introduced above to this effect will not be repeated. A double space will be used to mark off the portion of King Wan from that of his son.

Each hexagram consists of two of the trigrams of Fu-hsi, the lower being called 'the inner', and the one above 'the outer'. The lines, however, are numbered from one to six, commencing with the lowest. To denote the number of it and of the sixth line, the terms for 'commencing' and 'topmost' are used. The intermediate lines are simply 'second'

'third', etc. As the lines must be either whole or divided, technically called strong and weak, yang and yin, this distinction is indicated by the application to them of the numbers nine and six. All whole lines are nine, all divided lines, six.

Two explanations have been proposed of this application of these numbers. The *Khien* trigram, it is said, contains 3 strokes (☰), and the Khwan 6 (☷). But the yang contains the yin in itself, and its representative number will be 3 + 6 = 9, while the yin, not containing the yang, will only have its own number or 6. This explanation, entirely arbitrary, is now deservedly abandoned. The other is based on the use of the 'four Hsiang', or emblematic figures (⚌ the great or old yang, ⚎ the young yang, ⚍ the old yin, and ⚏ the young yin). To these are assigned (by what process is unimportant for our present purpose) the numbers 9, 8, 7, 6. They were 'the old yang', represented by 9, and 'the old yin', represented by 6, that, in the manipulation of the stalks to form new diagrams, determined the changes of figure; and so 9 and 6 came to be used as the names of a yang line and a yin line respectively. This explanation is now universally acquiesced in. The nomenclature of first nine, nine two, etc., or first six, six two, etc., however, is merely a jargon; and I have preferred to use, instead of it, in the translation, in order to describe the lines, the names 'undivided' and 'divided'.

I. Does King Wan ascribe four attributes here to Khien, or only two? According to Appendix IV, always by Chinese writers assigned to Confucius, he assigns four, corresponding to the principles of benevolence, righteousness, propriety, and knowledge in man's nature. Ku Hsi held that he assigned only two, and that we should translate, 'greatly penetrating', and 'requires to be correct and firm', two responses in divination. Up and down throughout the Text of the 64 hexagrams, we often find the characters thus coupled together. Both interpretations are possible. I have followed what is accepted as the view of Confucius. It would take pages to give a tithe of what has been written in justification of it, and to reconcile it with the other.

'The dragon' is the symbol employed by the duke of Kau to represent 'the superior man' and especially 'the great man', exhibiting the virtues or attributes characteristic of heaven. The creature's proper home is in the water, but it can disport itself on the land, and also fly and soar aloft. It has been from the earliest time the emblem with the Chinese of the highest dignity and wisdom, of sovereignty and sagehood, the combination of which constitutes 'the great man'. One emblem runs through the lines of many of the hexagrams as here.

But the dragon appears in the sixth line as going beyond the proper

limits. The ruling-sage has gone through all the sphere in which he is called on to display his attributes; it is time for him to relax. The line should not be always pulled tight; the bow should not be always kept drawn. The unchanging use of force will give occasion for repentance. The moral meaning found in the line is that 'the high shall be abased'.

The meaning given to the supernumerary paragraph is the opposite of that of paragraph 6. The 'host of dragons without their heads' would give us the next hexagram, or Khwan, made up of six divided lines. Force would have given place to submission, and haughtiness to humility; and the result would be good fortune. Such at least is the interpretation of the paragraph given in a narrative of the *Tso-Kwan* under 513 B.C.

II. The Khwan Hexagram

Khwan (represents) what is great and originating, penetrating, advantageous, correct and having the firmness of a mare. When the superior man (here intended) has to make any movement, if he take the initiative, he will go astray; if he follow, he will find his (proper) lord. The advantageousness will be seen in his getting friends in the south-west, and losing friends in the north-east. If he rest in correctness and firmness, there will be good fortune.

APPENDIX III

Section I

Chapter IX. 49. To heaven belongs (the number) 1; to earth, 2; to heaven, 3; to earth, 4; to heaven, 5; to earth, 6; to heaven, 7; to earth, 8; to heaven, 9; to earth, 10.

50. The numbers belonging to heaven are five, and those belonging to earth are (also) five. The numbers of these two series correspond to each other (in their fixed positions), and each one has another that may be considered its mate. The heavenly numbers amount to 25, and the earthly to 30. The numbers of heaven and earth together amount to 55. It is by these that the

changes and transformations are effected, and the spirit-like agencies kept in movement.

51. The numbers of the Great Expansion, (multiplied together), make 50, of which (only) 49 are used (in divination). (The stalks representing these) are divided into two heaps to represent the two (emblematic lines, or heaven and earth). One is then taken (from the heap on the right), and placed (between the little finger of the left hand and the next), that there may thus be symbolized the three (powers of heaven, earth and man). (The heaps on both sides) are manipulated by fours to represent the four seasons; and then the remainders are returned, and placed (between) the two middle fingers of the left hand, to represent the intercalary month. In five years there are two intercalations, and therefore there are two operations; and afterwards the whole process is repeated.

52. The numbers (required) for *Khien* (or the undivided line) amount to 216; those for Khwan (or the divided line), to 144. Together they are 360, corresponding to the days of the year.

53. The number produced by the lines in the two parts (of the Yi) amount to 11,520, corresponding to the number of all things.

54. Therefore by means of the four operations is the Yi completed. It takes 18 changes to form a hexagram.

55. (The formation of) the eight trigrams constitutes the small completion (of the Yi).

56. If we led on the diagrams and expanded them, if we prolonged each by the addition of the proper lines, then all events possible under the sky might have their representation.

57. (The diagrams) make manifest (by their appended explanations) the ways (of good and ill fortune), and show virtuous actions in their spiritual relations. In this way, by consulting them, we may receive an answer (to our doubts), and we may also by means of them assist the spiritual (power in its agency in nature and providence).

58. The Master said: 'He who knows the method of change and transformation may be said to know what is done by that spiritual (power).'

Chapter X. 59. In the Yi there are four things characteristic of the way of the sages. We should set the highest value on its

explanations to guide us in speaking; on its changes for (the initiation of) our movements; on its emblematic figures for (definite action as in) the construction of implements; and on its prognostications for our practice of divination.

60. Therefore, when a superior man is about to take action of a more private or of a public character, he asks (the Yi), making his inquiry in words. It receives his order, and the answer comes as the echo's response. Be the subject remote or near, mysterious or deep, he forthwith knows of what kind will be the coming result. (If the Yi) were not the most exquisite thing under heaven, would it be concerned in such an operation as this?

61. (The stalks) are manipulated by threes and fives to determine (one) change; they are laid on opposite sides, and placed one up, one down, to make sure of their numbers; and the (three necessary) changes are gone through with in this way, till they form the figures pertaining to heaven or to earth. Their numbers are exactly determined, and the emblems of (all things) under the sky are fixed.

*

In 49 and 50 'heaven' and 'earth' are used as we have seen *Khien* and Khwan are in paragraphs 30 and 34. Odd numbers belong to the strong or undivided line, which is symbolical of the active operation in nature, and the even numbers to the weak or divided line, symbolical of its inaction. The phraseology of the paragraphs, however, can only be understood by a reference to 'the river map', which has been given in the Introduction.

The map, as it appeared on the back of 'the dragon-horse', consisted of so many circles, and so many dark circular markings, the former, it was assumed, being of the Yang character, and the latter of the Yin. Fu-hsi for the circle substituted the strong or undivided line (——), and for the dark markings the weak or divided (— —). It will be seen that the Yang symbols are the 1, 3, 5, 7 and 9 circles, and the Yin are the 2, 4, 6, 8 and 10 circular markings, which is the pictorial delineation of paragraph 49. The only thing to be said upon it is that the arrangement of the five circles and ten circular markings is peculiar, and evidently devised 'for a purpose'. So far, however, as we know, no figure of the map was attempted till after the beginning of our twelfth century.

The same figure is supposed to illustrate what is said in paragraph 50: 'The numbers of the two series correspond to each other in their fixed

positions.' 1 and 2, and 3 and 4 certainly front each other, and perhaps 5 and 6; but 7 and 8, and 9 and 10 do not do so in the same way. It is said also that 'each has another that may be considered its mate'. So it is with 1 and 6, 2 and 7, 3 and 8, 4 and 9, but hardly with 5 and 10. Further, $1+3+5+7+9=25$; $2+4+6+8+10=30$; and $25+30=55$; all of which points are stated.

Paragraph 51 is intended to describe the process of divination in manipulating the stalks, but the description is confused by introducing into it the four seasons and the subject of intercalation, so as to be very difficult to understand.

Paragraph 52. The actual number of the undivided and divided lines in the hexagrams is the same, 192 of each. But the representative number of an undivided line is 9, and of a divided line 6. Now 9×4 (the number of the emblematic figures)$\times6$ (the lines of each hexagram)$=216$; and $6\times4\times6=144$. The sum of these products is 360, which was assumed, for the purpose of working the intercalation, as the standard length of the year.

Section II

Chapter II. 11. Anciently, when Pao-hsi had come to the rule of all under heaven, looking up he contemplated the brilliant forms exhibited in the sky, and looking down he surveyed the patterns shown on the earth. He contemplated the ornamental appearances of birds and beasts and the (different) suitabilities of the soil. Near at hand, in his own person, he found things for consideration, and the same at a distance, in things in general. On this he devised the eight trigrams, to show fully the attributes of the spirit-like and intelligent (operations working secretly), and to classify the qualities of the myriads of things.

12. He invented the making of nets of various kinds by knitting strings, both for hunting and fishing. The idea of this was taken, probably, from Li (the third trigram, and thirtieth hexagram).

13. On the death of Pao-hsi, there arose Shannang (in his place). He fashioned wood to form the share, and bent wood to make the plough-handle. The advantages of ploughing and weeding were then taught to all under heaven. The idea of this was taken, probably, from Yi (the forty-second hexagram).

14. He caused markets to be held at midday, thus bringing together all the people, and assembling in one place all their

wares. They made their exchanges and retired, every one having got what he wanted. The idea of this was taken, probably, from Shih Ho (the twenty-first hexagram).

16. They hollowed out trees to form canoes; they cut others long and thin to make oars. Thus arose the benefit of canoes and oars for the help of those who had no means of intercourse with others. They could now reach the most distant parts, and all under heaven were benefited. The idea of this was taken, probably, from Hwan (the fifty-ninth hexagram).

17. They used oxen (in carts) and yoked horses (to chariots), thus providing for the carriage of what was heavy, and for distant journeys – thereby benefiting all under the sky. The idea of this was taken, probably, from Sui (the seventeenth hexagram).

18. They made the (defence of the) double gates, and (the warning of) the clapper, as a preparation against the approach of marauding visitors. The idea of this was taken, probably, from Yu (the sixteenth hexagram).

19. They cut wood and fashioned it into pestles; they dug in the ground and formed mortars. Thus the myriads of the people received the benefit arising from the use of the pestle and mortar. The idea of this was taken, probably, from Hsiao Kwo (the sixty-second hexagram).

20. They bent wood by means of string so as to form bows, and sharpened wood so as to make arrows. This gave the benefit of bows and arrows, and served to produce everywhere a feeling of awe. The idea of this was taken, probably, from Khwei (the thirty-eighth hexagram).

23. In the highest antiquity, government was carried on successfully by the use of knotted cords (to preserve the memory of things). In subsequent ages the sages substituted for these written characters and bonds. By means of these (the doings of) all the officers could be regulated, and (the affairs of) all the people accurately examined. The idea of this was taken, probably, from Kwa (the forty-third hexagram).

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Chapter II, paragraphs 11.23, treats of the progress of civilization in China, and how the great men of antiquity who led the way in the various steps of that progress were guided by the Yi.

8

CHOU KUNG

(*c.* 1100 B.C.)

CHOU KUNG (also Chow Kung), the illustrious Duke of Chou, was a statesman, military genius and mathematician who flourished in China towards the end of the twelfth century B.C. Capable of prodigious bursts of energy, he was of inestimable service in establishing his brother Wu Wang as the first emperor of the Chou Dynasty. After Wu's death, he served as regent during the minority of the second emperor. The period of Chou Kung's activity in the early years of the Chou Dynasty remained a model and an ideal for succeeding generations. Though the Chou lasted almost a thousand years, no period or reign throughout this entire time has been described in terms of greater approbation by historians of the nation than the regency of the sagacious Chou Kung.

Chou Kung is accepted as the author of the first dialogue contained in the *Chou-Pi* (also *Chow-pei*, or *Chou-pei Suan-king*), one of the oldest of the ancient Chinese scientific treatises. Written in two parts, it deals principally with the calendrical problems, and thus with astronomy and mathematics. The first dialogue, as recorded, takes place between Chou Kung and a nobleman of the defeated Shang Dynasty. The questions and answers provide us with intimations of the level of mathematical learning attained in China by the twelfth century B.C. Clearly, we have evidence here of interest in and also knowledge of mensurational geometry, the principle of the Pythagorean theorem, elements of trigonometry basic to indirect measurement, and some instruments for astronomical measurements. The astronomical-mathematical material of the *Chou-Pi* formed the basis of Chinese works written more than a thousand years later.

CHOU-PI (CHOW-PI)

From Chinese Researches *by Alexander Wylie*

The *Chow-pi*, Thigh-bone of Chow.[1] The translation is given here of the first section, which forms an epitome of the whole:

'1. Formerly Chow Kung addressing Shang Kaou, said: I have heard it said my lord that you are famous at numbers; may I venture to ask you how the ancient Fo-hi established the degrees of the celestial sphere? There are no steps by which one may ascend the heavens, and it is impracticable to take a rule and measure the extent of the earth; I wish to ask then how he ascertained these numbers? 2. Shang Kaou replied: The art of numbering originates in the circle and quadrangle. 3. The circle is derived from the quadrangle. 4. The quadrangle originates in the right angle.[2] 5. The right angle originates in the multiplication of the nine digits. 6. Hence, separating a right angle into its component parts, if the base be equal to 3, and the altitude to 4, a line connecting the farther extremities will be 5. 7. Square the external dimensions, and half the amount will give the area of the triangle. 8. Add together all the sides, and the result will equal the sum of 3, 4 and 5. 9. The square of the hypotenuse being 25, is equal to the squares of the two short sides of the triangle. 10. Thus the means by which Yu restored order throughout the empire was by following out the principles of these numbers. 11. Chow Kung exclaimed: How truly great is the theory of numbers! May I ask what is the principle of the use of the rectangle? 12. Shang-kaou replied: The plane rectangle is formed by uninclined straight lines. 13. The direct rectangle is used for observing heights. 14. The reversed rectangle is used for fathoming depths. 15. The flat rectangle is used for ascertaining distances. 16. By the revolution of the rectangle the circle is formed. 17. By the junction of rectangles the square is formed. 18. The square per-

1. The name which at first sight appears a little *outré*, receives its explanation from the two characters Keu-koo used to designate the base and altitude of a triangle, and which mean originally the Leg and Thigh. It may call to mind also a similar device in our English term, Napier's Bones.

2. The term right angle is used here to designate the *keu-koo*, two short sides of a right-angled triangle.

tains to earth, the circle belongs to heaven, heaven being round and earth square. 19. The numbers of the square being the standard, the dimensions of the circle are deduced from the square. 20. The circular plate is employed to represent heaven; the celestial colours are blue and black, the terrestrial are yellow and red; the circular plate is formed according to the celestial numbers; it is blue and black outside, red and yellow inside, in order to represent the celestial and terrestrial stations. 21. Therefore he who understands the earth is a wise man; he who comprehends the heavens is a sage. This knowledge begins with the straight line; the straight line is a component part of the rectangle, and the numbers of the rectangle are applicable to the construction of all things. 22. Chow Kung exclaimed, Excellent indeed!'

9

CHANG TSANG

(d. 152 B.C.)

CHANG TSANG was a distinguished soldier, statesman and scholar of ancient China, who flourished in the second century B.C. He was the author of the greatest mathematical work of antiquity, the *K'iu-ch'ang Suan-shu*, or *Kew-chang-swan-shuh* (Arithmetical Rules in Nine Sections), a restoration of a vastly older work of the same name. Chang began his career in the service of his country as a civil servant. Later, he fought brilliantly in the wars of the first Han emperor, Liu Pang, gaining great fame for his remarkable military exploits. During the ensuing peaceful years, Chang served again as an able and loyal official of the government, rising to the position of chief minister in 176 B.C., an office which he held until his death in 152 B.C. During his tenure as chief minister, Chang's fame increased through his distinguished service in the financial administration of the government. He was held in the highest esteem, however, for his learning in the fields of mathematics, astronomy and astrology. Chang is said to have been more than one hundred years old when he died.

Chang Tsang's mathematical work, the *K'iu-ch'ang Suan-shu*, was based on fragments of the earlier *K'iu-ch-'ang* which had escaped the book-burning ordered by Ch'in Shih Huang-ti (Ts'in Shih Huang-ti) in the latter part of the third century B.C. At that time, despite the knowledge that 400 scholars had been buried alive for protesting against the book-burning, and despite the continuing threat of dire penalties for failure to obey the order for the destruction of the proscribed books, there were individuals who secretly clung to and preserved some fragments of the ancient books. When the ban on these books was lifted by the second emperor of the Han, a diligent search was made for any works that may have survived the conflagration. Gradually, piece by piece, remnants of the old works came out of hiding, and in the light of these fragments, Chang Tsang reconstructed the ancient text. The rules of arithmetic contained in the old text were the mathematical rules taught by the Royal Tutor of the Chou dynasty at a time which was

ancient even in Chang's day. Tradition places the origin of the early work in the third millennium B.C. The text of Chang Tsang's work, as we possess it, was compiled from items extracted from the encyclopaedia, *Yung lo ta tien*, which was written in the fifteenth century. Edited and re-edited with extreme care, Chang Tsang's ancient mathematical classic has been published many times.

Chang Tsang's *K'iu-ch'ang Suan-shu* contains 246 problems classified into nine groups or sections. Each section is headed by a stanza giving an accurate and concisely stated general rule for solving the problems in the section. Among the topics involved in the problems, are proportion, the extraction of square and cube roots, partnership, plane mensuration, solid mensuration, and the right triangle. Frequent use is made of the rule of false position. The *K'iu-ch'ang Suan-shu* is the earliest mathematical work in which mention is made of negative numbers.

ARITHMETICAL RULES IN NINE SECTIONS

From Chinese Researches *by Alexander Wylie*

From Chang Tsang *Kew-chang-swan-shuh*, 'Arithmetical Rules of the Nine Sections.' – Suppose there are a number of rabbits and pheasants confined in a cage, in all 35 heads and 94 feet: required the number of each? Ans. 23 pheasants, 12 rabbits.

A number of men bought a number of articles, neither of which are known; it is only known that if each man paid 8 cash, there would be a surplus of 3 cash; and if each man paid 7 cash, there would be a deficiency of 4 cash; required the respective numbers? Ans. 7 men, 53 articles.

. . . – Equations. This section is remarkable as containing an exposition of the use of the terms *ching*, 'plus' and *foo*, 'minus'. In a series of 18 problems it gives the method of ascertaining the value of unknown quantities from certain conditions of combination, depending on the number of terms in the equation. The following is one of the simplest examples: If 5 oxen and 2 sheep cost 10 taels of gold, and 2 oxen and 5 sheep cost 8 taels, what are the prices of the oxen and sheep respectively? Ans. Each ox 1 tael and 13 twenty-first parts; each sheep, 20 twenty-first parts of a tael.

In the following examples let a represent the base, b the altitude and c the hypotenuse.[1] 1. Given the difference of altitude and hypotenuse, and the base; to find the hypotenuse.

$$(a^2-\overline{c-b}^2)\div 2\overline{c-b} \text{ Ex.}$$

There is a pool 10 feet square, with a reed growing in the centre, which rises a foot above the surface; when drawn towards the shore it reaches exactly to the brink of the pool; what is the depth of water? Ans. 12 feet. 2. Another method.

$$(a^2+\overline{c-b}^2)\div 2\overline{c-b} \text{ Ex.}$$

On opening a two-leaved door, when the inner edge of the leaves are a foot from the door frame, there is an opening of 2 inches between the leaves; what is the width of the door? Ans. Each leaf $72\frac{1}{2}$ inches. 3. Another method.

$$a^2\div\overline{c-b}+c-b \text{ Ex.}$$

A chain suspended from an upright post has a length of 2 feet lying on the ground, and on being drawn out to its full length, so as just to touch the ground, the end is found to be 8 feet from the post; what is the length of the chain? Ans. 17 feet. . . . 5. Given the sum of the altitude and hypotenuse, and the base, to find the altitude.

$$\frac{b+c-\overline{a^2\div b+c}}{2} \text{ Ex.}$$

There is a bamboo 10 feet high, the upper end of which being broken down on reaching the ground, the tip is just 3 feet from the stem; what is the height to the break? Ans. feet 4 and 11 twentieths.

*

What is the diameter of the largest circle that can be inscribed within a right-angled triangle, the two short sides of which are respectively 8 and 15. Ans. 6.

*

1. Algebraic symbols are used here for the sake of brevity; it is not meant to imply that these are in the original.

CHANG TSANG

From The Development of Mathematics in China and Japan
by Yoshio Mikami

'While a good runner walks 100 paces (or a), a bad runner goes 60 paces (or b). Now the latter goes 100 paces (or c) in advance of the former, who then pursues the other. In how many paces will the two come together?'

The answer is: $a \times c \div (a-b) = 250$ paces.

'A hare runs 100 paces (a) ahead of a dog. The latter pursues the former for 250 paces (b), when the two are 30 paces (c) apart. In how many further paces will the dog overtake the hare?'

The answer is: $cb \div (a-c) = 107\frac{1}{7}$ paces.

'When buying things in companionship, if each gives 8 pieces, the surplus is 3; if each gives 7, the deficiency is 4, It is required to know the number of persons and the price of the things bought.'

'When buying hens in companionship, if each gives 9, then 11 will be surplus; and if each gives 6, then 16 will be the deficiency. What will be the number of persons and the price of the hens?'

These problems are equivalents to the solution of the equation

$$y = ax - b \text{ and } y = a'x + b'.$$

The rule for solution is given as follows:

'Arrange the rates (a and a') forwarded by the partners in buying things. What surpasses (b), or is deficient (b'), be each arranged below these rates, and then cross multiply with them. Add the products together, and one gets the *shih*. The surplus and deficiency being added, make the *fa*. If fractional, first make both members equidenominated. Then the *shih* and the *fa* being divided by the difference of the rates, the quotients represent the price of the things bought and the number of persons, respectively.'

According to this rule we first form the arrangement (1), from which by cross multiplication follows (2); and then by addition we have (3). Thus

$$(1)\ \begin{matrix} a & a' \\ b & b' \end{matrix}, \qquad (2)\ \begin{matrix} ab' & a'b \\ b & b' \end{matrix}, \qquad (3)\ \begin{matrix} ab'+a'b \\ b+b' \end{matrix}.$$

And now we have to take price $= \dfrac{a'b+ab'}{a-a'}$, persons $= \dfrac{b+b}{a-a'}$.

A second rule for the same case, which is this: 'Add the surplus and deficiency, which makes the *shih*. The difference of the rates is the *fa*. The *shih*, divided by the *fa*, gives the number of persons. Multiply it by the rates and subtract the surplus or add the deficiency, when we get the price of the things.'

'Of two water-weeds, the one grows 3 feet and the other one foot on the first day. The growth of the first becomes every day half of that of the preceding day, while the other grows twice as much as on the day before. In how many days will the two grow to equal heights?'

The answer is $2\frac{6}{13}$ days, when both grow to the same height of 4 feet and $8\frac{6}{13}$ decimal parts.

'There are three classes of corn, of which three bundles of the first class, two of the second and one of the third make 39 measures. Two of the first, three of the second and one of the third make 34 measures. And one of the first, two of the second and three of the third make 26 measures. How many measures of grain are contained in one bundle of each class?'

'Rule. Arrange the 3, 2 and 1 bundles of the three classes and the 39 measures of their grains at the right. Arrange other conditions at the middle and at the left.'

The arrangement then takes the form shown in the accompanying diagram.

1	2	3	1st class
2	3	2	2nd ,,
3	1	1	3rd ,,
26	34	39	measures

The text proceeds 'With the first class in the right column multiply currently the middle column, and directly leave out.'

This means to subtract the terms in the right column as often as possible from the corresponding terms of the middle column thus multiplied.

1	0	3	1st class
2	5	2	2nd „
3	1	1	3rd „
26	24	39	measures

'Again multiply the next, and directly leave out.'

0	0	3	1st class
4	5	2	2nd „
8	1	1	3rd „
39	24	39	measures

'Then with what remains of the second class in the middle column, directly leave out.'

That is, the 2nd class from the left column is to be eliminated be applying the process as above described. The result is as shown here.

36	3rd class
99	measures

'Of the quantities that do not vanish, make the upper the *fa*, the divisor, and the lower the *shih*, the dividend, i.e. the dividend for the 3rd class.

'To find the 2nd class, with the divisor multiply the measure in the middle column and leave out of it the dividend for the third class. The remainder, being divided by the number of bundles of the second class, gives the dividend for the 2nd class.

'To find the 1st class, also with the divisor multiply the measures in the right column and leave out from it the dividends for the

third and second classes. The remainder being divided by the number of bundles of the 1st class, gives the dividend for the 1st class.

'Divide the dividends of the three classes by the divisor, and we get their respective measures.'

The answer is given as $9\frac{1}{4}$, $4\frac{1}{4}$ and $2\frac{3}{4}$ measures of grain, respectively.

'There are three kinds of corn. The grains contained in two, three and four bundles, respectively, of these three classes of corn, are not sufficient to make a whole measure. If, however, we add to them one bundle of the 2nd, 3rd and 1st classes, respectively, then the grains would become full one measure in each case. How many measures of grain does then each one bundle of the different classes contain?'

The answer is given as $\frac{9}{25}$, $\frac{7}{25}$, $\frac{4}{25}$ measures of the three classes.

*

The 'Nine Sections' explains as to the treatment of the positive and negative quantities thus:

'When the equi-named (or equally signed) quantities are to be subtracted and the different named are to be added (in their absolute values), if a positive quantity has no opponent, make it negative; and if a negative has no opponent, make it positive. When the different named are to be subtracted and the same named are to be added (in absolute values), if a positive quantity has no opponent, make it positive; and if a negative has no opponent, make it negative.'

10

R. NEHEMIAH

(*c.* A.D. 150)

R. NEHEMIAH, a Hebrew rabbi, scholar and teacher, who lived in Palestine during the second century of the Common Era, was the author of the *Mishnat ha-Middot*, the earliest Hebrew treatise on geometry known to us. Nehemiah was one of the *Tannaim*, that is, one of the spiritual and intellectual leaders of Palestine of the first and second centuries, whose teachings, when compiled and codified, were to become the basis of a very extensive future rabbinical literature. A pupil of the famous R. Akiba, and a contemporary of Meir and Judah b. Ilai, Nehemiah belonged to the 'Fourth Generation' of *Tannaim*, a group of scholars and teachers who were the source of religious and civil authority in Palestine from about 140 to 165. Nehemiah's maxims together with his explanations and supplementary notes, further enriched and elucidated by succeeding writers, constitute the foundation of the *Tosefta*, the rabbinical interpretation of Biblical law, as it has come down to us. The substance of his teachings, and the method of full and detailed presentation preferred by him, reveal Nehemiah as a zealous and untiring force in advancing the study and understanding of the Bible-verse. Nehemiah was also a mathematician. In this capacity, he was concerned with the determination of the calendar and with the study of the measures of the Ark and of the Tabernacle.

The old *Tannaitic* treatise, the *Mishnat ha-Middot*, became known in modern times through the translation and publication of two manuscripts. The first manuscript, Cod. Hebr. 36, in the Munich Library, containing five chapters dealing with the elements of plane and solid geometry, appeared in a German translation in 1862. The other, MS. Heb. c. 18 of the Bodleian Library, more recently found, is a fragment consisting of two leaves which contain part of the first, second and fifth chapters already known and, in addition, the beginning of a sixth chapter, hitherto unknown. The fragment, on vellum and in Syr. Rabb. characters (sixth chapter), contains explanations of the size and the construction of the Tabernacle. The two manuscripts together have

made it clear that the first five chapters of the *Mishnat ha-Middot* form a brief treatise on geometry which was introductory to a work elucidating the details of the construction and the measurements of the Tabernacle. The geometrical principles presented in the *Mishnat ha-Middot* were known for the most part to the mathematicians of Nehemiah's day. His own contributions were introduced in the manner of the ancient authors, by the words, 'Nehemiah says'. There is considerable evidence in the *Mishnat ha-Middot* of Nehemiah's knowledge of Greek mathematics on the one hand, and on the other of the influence of the old Hebrew work on Persian – Hindu – Arabic mathematics.

MISHNAT HA-MIDDOT

Translation and notes by Solomon Gandz

CHAPTER I

§ 1. There are four ways to grasp the area, namely: the quadrilateral, the trilateral,[1] the circle, and the bow-figure.[2–3] The rule is as follows: the second figure is the half of the first one, and the fourth is the half of the third one.[4] All the rest of the figures are closely connected with them, respectively as the hook with the ring.

1. Not 'quadrangle' and 'triangle', see Gandz, *The Origin of Angle Geometry*, p. 473 ff. *Isis*, v. 12.

2. I.e. segment.

3. Euclid, *Elements*, I, def. 15–29 (cf. also Hero, *Opera Omnia*, IV, p. 180), speaks of five forms of figures in such an order that the boundary lines are increasing in number from one to *n* lines, to wit: 1. the circle, 2. the semi-circle, 3. the trilateral, 4. the quadrilateral, and 5. the multilateral. Our treatise brings only four figures, and in the converse order. It is interesting to note that Ganesa (*c.* A.D. 1520), the Hindu commentator to the Lilavati of Bhascara (*c.* A.D. 1150), has the same order and number as the *Mishnat ha-Middot*. He says: 'Plane figure is four-fold: triangle, quadrangle, circle and bow'; see Colebrooke, p. 58, note. Although Ganesa lived as late as *c.* 1520, he must have followed some old tradition and source common to Hindu mathematics and *Mishnat ha-Middot*; see Gandz, *The Origin of Angle Geometry*, p. 477.

4. So he understands here under 'bow-figure' the semi-circle, while in the following – I, 5; V, 4 seq. – any segment is meant. Dr Neugebauer calls the attention to the fact that in Babylonian mathematics, too, no distinction is made between semi-circle and segment.

§ 2. The quadrilateral has three aspects:[1] the side, the thread[2] and the roof.[3] Which is the side? That holds fast the walls of the roof:[4] for it is written:[5] *Foursquare shall be the altar.*[6] And the thread (diagonal) which is? That cuts[7] through from one angle to the other angle, *from*[8] *one corner to the other corner.* And that

1. Literally 'face', then 'view', 'viewpoint'. The meaning is: it can be considered, investigated from three viewpoints; there are three elements to be examined in the quadrilateral.

2. Khawt, = 'thread, cord, rope', then 'line'. The word 'line' comes from the Latin *linea*; this comes from *linum* 'linen, rope'; see Tropfke, IV, 2nd ed., p. 38. The word is commonly used for 'diagonal' and 'diameter' in this treatise; see I, 3, 4; II, 3, 8; III, 4; V, 1, 2, 3, 7. The Chinese use a word denoting 'rope' for the diagonal; see Cantor, I, 4th ed., pp. 679–80. The same is the case with the Hindus; see Colebrooke, p. 59, note 2; Smith, *History of Mathematics*, I, p. 98. On the use of the same term for diagonal and diameter see below note 7. Cf. also Gandz, *Die Harpendonapten*, p. 272 f.

3. 'Roof', 'surface', 'area'; compare these words with the Arabic *sath*, originally 'flat roof'; see the writer's article 'Origin of the term Root' in *The American Mathematical Monthly*, vol. 33, p. 263, note 2. This Arabic *sath*, that is a translation of the Hebrew *gag*, was later retranslated by the Hebrew *shetah*, and used by Savasorda and Israeli, etc.

4. The quadrilateral is imagined as the flat roof of a building, and its sides as holding fast the walls of the building.

5. Exodus, 27, 1.

6. The reason for the quotation of this verse is not quite clear. The author apparently understands under *m'ruvet* a square; see p. 192, note 1, and p. 190, note 3. He cites this verse the altar shall be a square and thinks also of the continuation in v. 7: And the staves shall be upon the two sides of the altar, where the word tzeleh is used for the side of the square altar. This way of quoting only one part of a verse, or one verse of a section and thinking also of the following parts or verses is usual in the Midrashim. They rely upon it that the reader will look up the whole verse or section.

7. The line that cuts the figure running from one corner to the other corner. It is worth while to note that the ancient Greek terminology had no special term for the diagonal, but called it a 'line stretched from one corner to the other corner'. The Sophists created the term meaning 'diameter', using it promiscuously for the diagonal and the diameter of the circle. Hero introduced the term which stands for 'diagonal'. I refer to Hebrew–Aramaic *zalaf* = 'the dropping, dripping of the rain', and remark that strings of the raindrops usually come diagonally. This is corroborated by the Arabic *qutr* = 'dropping rain, diagonal'. The height is the falling stone, or plummet, because it falls perpendicularly. The diagonal is the string of the raindrops falling diagonally; see p. 192, note 5.

8. Exodus 26, 28; 36, 33. The fact is that these verses refer to the two

is the greatest [straight line] in the length of the roof.[1] And the roof itself is the area.

§ 3. And the trilateral has four aspects: the side-pair,[2] the basis, the pillar, and the roof. Which is the side-pair? These are the two straight lines to the right and left side; for it is written:[3] *For thou shalt spread abroad to the right and the left.* And the basis? That is upon which the two sides are established; for it is written:[4] *Upon which the house is established.* And the pillar? That is the common thread running down[5] to the basis from

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corners of one side and not to the opposite corners of the diagonal. These verses, as many of the following passages of the Scriptures, seem to be adduced only to show that the terms or similar words occur also in the Bible. It is in the style of the old Mishnah to connect the law with the Bible in any way. Cf. Zunz, Gesammelte Schriften, III, p. 243, who says concerning the Baraitha di Semuel: '*Bei einzelnen Satzen werden, gleichsam um dem Gedachtnisse des Lesers zu Hilfe zu kommen, Bibelstellen angefugt*'.

1. The author thinks only of the square or rectangle where both diagonals are equal and the longest lines. In the rhombus this is not the case. Cf. also p. 190, note 3.

2. Comp. Schapira, p. 15, note 1. It is clear that the isosceles is here meant, having two equal sides or legs. In *The Rhind Mathematical Papyrus*, too, the figures show that only isosceles were taken into consideration; see Gandz, *The Origin of Angle Geometry*, p. 470; Cantor 1^4, p. 111; Chace, A. B., *The Rhind Mathematical Papyrus*, p. 36. Peet, pp. 91–2, brings a full discussion of the subject without reaching a decision. But Struve, pp. 147–152, convincingly proves the isosceles character of the triangles. Struve's conclusion was adopted by Neugebauer and corroborated by a Babylonian parallel; see QS. B 1, p. 416. Much later the scalene was considered; hence the name scalene = 'limping'. The normal triangle was that with two even legs.

3. Isaiah, 54, 3.

4. Judges, 16, 26, 29. The Hebrew thought of the triangle as of an erected isosceles representing the tent with two equal pillars established upon the base; hence the name *keveh* = 'basis', and *tzeleh* = 'wall, side'. 'To the Egyptian mind', however, 'triangles mostly lie flat on the ground'; cf. Chace, R. M. P., p. 37, note 1, quoting Gunn, and Struve, *Quellen u. Studien*, A. 1, p. 154. Therefore the Egyptian terms *tpr* = 'mouth' for the base, and *mryt* = 'rim, edge', 'border' for the side or rather for the height; see Struve, p. 154 seq.; Neugebauer, QS, B. 1, p. 416, note 14, and p. 190, notes 2 and 3.

5. *Yerekh* = Arabic *warad* = Assyrian *aradu*, or *waradu* = 'to come down, fall down'. Hence the Assyrian term *urdu*, *urdam*, meaning 'the coming down' for the arrow of the segment and for the height; see Neugebauer

between the two sides. And it is in the corner,[1] *for the corners of the tabernacle.* And the roof itself is the area.

§ 4. The circle has three aspects: the circumference, the thread and the roof. Which is the circumference? That is the rope surrounding the circle; for it is written:[2] *And a rope of thirty cubits encompassed it round about.* And the thread? That is the straight-line from brim to brim: for it is written:[3] *From brim to brim.* And the roof itself is the area.

§ 5. And the bow figure[4] has four aspects: the bow, the chord,[5] the arrow[6] and the roof. Which is the bow? That is a portion of

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QS, B. 1, p. 82 Archiv f. Orientforschung 7, p. 91 nr. 3 (1931). The Egyptian term for height is also *prmws* = 'what comes out, or falls down from the window'; see Struve, QS, A. 1, p. 136; Neugebauer, QS, B. 1, p. 436, note 72. The Greek term for the perpendicular is equivalent to 'let fall, let down'. Al-Khowarizmi, in his geometry, calls the foot point of the perpendicular: 'The point where the stone falls down.' They all think of the 'plumb-line' or 'plummet' that falls perpendicularly. The diagonal is the *ziliptu* = *qutr*, 'the falling raindrops'; see p. 191, note 7.

1. Add: 'For it is written.' The verse is in Exodus, 26, 23.

2. I Kings, 7, 23.

3. I Kings, 7, 23: 'And he made the molten sea of ten cubits from brim to brim.' The circle is thought of as lying flat on the ground; see further V, 1, 3. Like the molten sea and the Egyptian triangles, see above note 7, it has a brim. See also Heath, *Euclid* I, p. 184, citing Proclus' quotation from the oracles: 'the centre from which all (lines extending) as far as the *rim* are equal'.

4. Literally: 'the bow-like, arched, vaulted'.

5. Or 'string.'; see also II, 4, and V, 5–7. The Hindus have for the 'chord' the term *jya*, or *jiva*, which came to be the origin of our *sinus*. In the geometry of al-Khowarizmi, we have still the term *watar* corresponding to the Hebrew *yeter*. Later on the Arabs transliterated the Hindu *jiva* as *jiba*, which was mistaken for the Arabic word *jaib*, 'bosom, fold', and then translated, first by Gerhard of Cremona (*c.* 1150), as *sinus*. This origin of the term *sinus* was first discovered by Munk in *Journal Asiatique*, 1863, p. 478, note. The most elaborate report on the subject and its literature is given by Tropfke, *Geschichte der Elementar-Mathematik*, V², pp. 31–3. Cf. also Braunmuhl, *Geschichte der Trigonometrie*, I, p. 49. Ruska, in *Zeitschrift fur Mathematik und Physik*, 1895, Literarisch-Historischer Teil, p. 127, says: '*Es scheint dass der ganze Wortkomplex Bogen, Sehne, Pfeil aus der indischen Trigonometrie herruhrt.* Here, however, we find this terminology in the Hebrew geometry of *c.* A.D. 150.

6. It is quite natural that the primitive mind borrowed his terminology

the circle; for it is written:[1] *Like the appearance of the bow that is in the cloud.* The chord? That is which holds fast[2] the mouth[3] of the bow; for it is written:[4] *The bent bow.* And the arrow? That is the straight line from the middle of the bow to the middle of the chord;[5] for it is written:[6] *They fix their arrows upon the chord.* And the roof itself is the area.

§ 6. How does one measure the area in numbers?[7] You figure one upon one[8] that is the area,[9] and it is one cubit upon one cubit. Thus, if a roof[10] has equal sides and angles you count them from each side, [and if it has one cubit to each side then the whole roof is one]. And if the plane[11] has two [cubits] from each side and

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for the segment from the hunter's bow. (The hunter's bow is also believed by some scholars to have suggested the first musical instrument, the stringed instrument; see L. Thorndike, *History of Civilization*, p. 24 and cf. also Wreszinski, W., *Atlas zur altagyptischen Kulturgeschichte*, plates No. 80, 81, representing Egyptian artisans of the fifteenth century B.C. in the procedure of making bows; ib. also the main literature for the history of the bow). The same terminology was also used in the Talmudic sources, see Babli Erubin 55a (cf. also Tosefta IV, 3; ed. Zuckermandel, VI, 1; Yer. V, 1).

1. Ezekiel, 1, 28.

2. Compare above, §2 'the side ... that holds fast the walls of the roof ...'.

3. The Egyptians called the base of an isosceles triangle and of an isosceles trapezium *tpr* = 'the mouth'; see Peet, *The Rhind Mathematical Papyrus*, p. 91, Struve, p. 152, and Neugebauer, QS, B. 1, pp. 416, 419; the Hindus named the upper side of a quadrangle *mukha*, or *vadana*, = 'opening, mouth'; see Colebrooke, p. 72, note 4, and p. 307, § 36, quoted by Cantor, I^4, p. 647.

4. Isaiah 21, 15. There is no mention of the chord here. The meaning is, perhaps, that the bow is bent by the chord that holds fast the two ends.

5. The arrow is the height of the segment. We may say that M. ha M. here teaches to construct a perpendicular by drawing a straight line from the middle of the arc to the middle of the chord, which is essentially like Euclid, *Elements* I, 12 (see Heath, I, p. 271). Only, that Euclid also proves it.

6. Psalms, 11, 2. This verse could have been used for all the three terms. The beginning reads: 'For, lo, the wicked bend the bow.'

7. The meaning is: How do you compute the area when the numbers of the sides are given?

8. See Gandz, *Terminology of Multiplication*, p. 261, notes 74–7.

9. i.e. the unit of the area. 10. Surface or plane; see note 7 above.

11. טבלא = *tabula* = τάβλα = 'board, tablet'; it is here used for surface and plane like the term 'roof'. Usually the tablet was square, טבלא מרובעת; see *Mishna Erubin*, IV, 8; V, 1.

the angles are equal, the area contains four times the measure of the unit, which is one cubit upon one cubit. And if it has three from each side, then [the area] is nine times the measure of the unit, and so is four upon four and five upon five. From now on go ahead and figure according to this measure[1] upwards.[2]

§ 7. Those[3] less than a unit you divide as follows: One cubit[4] into two threads, cutting each other in the middle, [running] from the right side to the left side and also from the top to the bottom. The roof is thus divided into four sections and you find half a cubit upon half a cubit [for each side] and the area itself is the fourth part of a cubit, which is the fourth of a square cubit. So is a third upon a third, and a fifth upon a fifth, with equal and unequal [sides]. From now on go ahead and figure with fractions according to this measure[5] downwards.

§ 8. They already said:[6] Half upon half is a fourth [part], and

1. Or 'method'.

2. This somewhat elaborate definition of the unit of the area is very important. It is lacking in Euclid; see Gutmann's note to Hibbur ha Meshiha, p. 22, § 42. Its first appearance is here, from where al-Khowarizmi took it; §1 of his geometry.

3. § 7 gives the geometric demonstration for the rule of the fractions. The same demonstration is also found in the Hibbur ha Meshiha, § 43, p. 23 for the integers, i.e. to prove that the double side gives four times the square of the unit, etc. In § 8 the arithmetical form of the rule is given and an arithmetical demonstration by the change of the position in the decimal order. Then comes in § 9 the general rule corresponding to the multiplication of fractions, and finally II, 1 gives the general rule for getting the area of rectangles by multiplying length into breadth. Contrary to his habit, the author of the *Mishnat ha-Middot* gives here demonstrations and proofs in order to bring home the very important idea and right conception of the areal content. The error in those times was widespread that the same perimeter gives the same area, and plain people could not understand how the square of the double side can give more than the double of the area; see the interesting examples quoted by Cantor I[4], pp. 172–3, and also the wonderful passage in Plato's *Dialogi*, ed. Hermann, III (Lipsiae 1851), pp. 333–9; Engl. translation by B. Jowett, II, 3rd ed., pp. 41–6. There Socrates proves to Menon's boy that the double square does not develop from the double line, as the boy originally thought and answered, but from the diagonal of the first square. The proof is by drawing and demonstration like here. See also Gandz, *Terms relating to Area*, p. 85.

4. Is to be bisected. 5. Or 'method'.

6. i.e. it is an old tradition, or rule.

so is a third upon a third a ninth [part]. With these and their similars, with equal and unequal [sides]. . . . You figure as follows: Ten upon ten are amounting to a hundred. The half of ten is five. Five times five are twenty and five, and this is the fourth of 100. And the position of the 10 is in the 1, and the position of the 100 in the 10, and of the 1000 in the 100. From now on go ahead and figure with the fractions according to the measure of the units.[1] But with the units it is[2] increasing and with the fractions it is decreasing.

§ 9. The rule is as follows: Half upon a half is the half of the half. And a third upon a third is the third of the third. So is the half upon a third the half of the third. And so is a fourth upon a third the fourth of the third. With these and their similars with equal and unequal [sides].

CHAPTER II

§ 1. If someone wants to measure quadrilateral fields, with equal or unequal sides, let him multiply length upon breadth, and the result[3] from both is the area.

§ 2. If a trilateral, with equal or unequal sides, let him multiply the pillar[4] in the half of the basis, and the result from both is the area. But there are many other methods for that.[5]

§ 3. How is it with the circle? Let him multiply the thread[6] into itself, and throw away[7] from it one-seventh and the half of a seventh; the rest is the area. For instance: The thread is 7 long,[8]

1. Integers.

2. The product and the square content.

3. Literally: *oleh* = to go up, 'climb, ascend, reach': cf. the Latin *resilio* 'to spring back, rebound', from which the term 'result' comes, and the Arabic *fama balagha* 'to reach, amount'.

4. Literally: *ha-omed* = 'the upright standing'.

5. For the finding of the area of the triangle; this may allude to the methods set forth in Chapter IV.

6. The diameter; see p. 191, notes 2 and 7.

7. This is the usual term for subtraction.

8. Literally: 'extended to seven'; the length of diameter is usually given as seven, that the circumference ($c. = \frac{2}{7}\frac{2}{7}d$) might give a whole number (22): see V, 3.

its multiplication [into itself] is 49; a seventh and half a seventh [of it] is $10\frac{1}{2}$; the area is thus $38\frac{1}{2}$.[1]

§ 4. How is it with the bow [figure]?[2] Let him add the arrow to the chord, [take] both together and multiply them into the half of the arrow and put them aside;[3] and again let him take the half of the chord, multiply it into itself and divide it upon 14 and the result let him add to the one[4] standing [aside]; the resulting [sum] is the area.[5] But there are other methods in that.[6]

1. The formula given here for the area of the circle is

$$A = d^2 - \tfrac{1}{7}d^2 - \tfrac{1}{14}d^2 = \tfrac{11}{14}d^2.$$

This is based, of course, upon the formula $A = \frac{1}{2}$ *circumference* multiplied by the radius, or $\frac{c}{2}\cdot\frac{d}{2}$, and upon the computation of $\pi = 3\frac{1}{7}$. We have then $\pi r^2 = \frac{22}{7}\cdot\frac{d^2}{4} = \frac{11}{14}d^2$. The same formula is given Hero, IV, p. 387, in the name of Archimedes as follows: eleven squares erected upon the diameter are equal to fourteen areas of the circle, $11d^2 = 14A$; $A = \frac{11}{14}d^2$; see also ibid., III, p. 65, and IV, p. 182–3. In Hero, ibid., IV, p. 332, Euclid is quoted to have given the area of the circle as $d^2 - \frac{1}{7}d^2 - \frac{1}{14}d^2$, exactly like the *Mishnat ha-Middot*, Hero, ibid., uses also ἐκβάλλω 'to throw away' the seventh and fourteenth part of d^2 and the usual numbers are $d = 7$, $c = 22$. As a matter of fact, however, Euclid in his *Elements*, or other writings known to us, never gives the computation of the area of the circle, nor of other areas or volumes. The formula $A = d^2 - \frac{1}{7}d^2 - \frac{1}{14}d^2$ is also in *Hibbur ha Meshiha*, § 96, p. 61; in the geometry of al-Khowarizmi, § 4 *in fine*, and of al-Karkhi (*c.* 1020), *Kafi fil Hisab*, II, p. 23.

2. The segment.

3. As one addend.

4. Addend.

5. If we call the chord c, the arrow h, and F the area, then the formula is $F = (c+h)\frac{h}{2} + \frac{1}{14}\left(\frac{c}{2}\right)^2$. For the semicircle the formula is accurate. $c + h = 3r$; hence the formula is $\frac{3r^2}{2} + \frac{1}{14}r^2 = F$, which is equal to $\frac{\pi r^2}{2}$, if we take $\pi = 3\frac{1}{14}$. For the other segments the formula is accurate to two decimals; see Schapira, p. 20, note 7. If we take $\pi = 3$, then the formula is reduced to $(c+h)\frac{h}{2}$, in which form it was known to the Chinese (*c.* 176 B.C.) and to the Hindu Mahavira (*c.* 850); see Smith, *History of Mathematics*, I, pp. 139, 164. Hero of Alexandria refers to this inaccurate formula of the ancients and improves it by the addition of $\frac{1}{14}\left(\frac{c}{2}\right)^2$ to $(c+h)\frac{h}{2}$; see Hero, III, p. 72, seq., p. 80, line

§ 5. If one wants to measure a square solid[1] in numbers, with equal or unequal sides, having length, breadth and thickness, six faces,[2] 'each one had six wings',[3] let him multiply length into breadth into depth,[4] and the result from all three is the volume[5] of the solid, and that is also the body.

§ 6. If it was circular,[6] or trilateral,[7] or of any number of sides, provided that its depth was straight and fair,[8] then measure its roof[9] according to the [above] mentioned measure, and you will ascertain the area; and the result you will multiply into the depth. And the [result] is the bulk of the body.

§ 7. And the decreasing one[10] [like the ethrog],[11] with a sharp top

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12, p. 83; IV, p. 357 seq. and p. 363 seq.; V, pp. 185-91; Cantor, I[4], p. 379. This formula is also found in Calumella's (*c.* 60 C.E.) works; see Cantor, ibid., p. 549, but not in Savasorda's nor in al-Khowarizmi's geometry. In Babylonian mathematics we find computations of chord, arrow and diameter, but not of the arc; see Struve-Neugebauer, QS., B. 1, pp. 90–2.

6. For the finding of the area of the segment, see V, 5, 6.

1. A body with quadrilateral surfaces and right angles. *Gov*, or *Govoh* = 'back', then *Govyah* = *Guf* = body.

2. Surfaces. Hero, def. 105, quoted by Heath, *Euclid*, III, p. 269, speaks of prisms having six faces.

3. Isaiah 6, 2.

4. See Gandz, *The Origin of Angle Geometry*, § 15, p. 469.

5. Here and in the following, §§ 6, 7, 9, 10, the term *mishikhah* is used for the volume of a body, too.

6. A cylinder.

7. A prism.

8. B. has *Yesher v'shav*, 'straight and even'. The meaning is that base and top are parallel. But conception and term of parallelism are still missing in M. ha-M.; see Gandz, *The Origin of Angle Geometry*, p. 466.

9. The area of the top or bottom.

10. *M'shokh*, 'drawn together, contracted', as subsequently explained, a solid having a flat, plane base and tapering, being contracted to a point or acute top; cf. Hero, IV, p. 54 seq.: A cone is a solid figure having a circle for its base that is *contracted* to a point.'

11. MS. B.: has *v'ham'shokh k'ethrog* and the word *ethrog* occurs there again in § 9. The word is known from the Talmudic literature. The *ethrog* is a kind of a citron which is nearly cone-shaped. It is prescribed for the religious observances of the Feast of Tabernacles. Similarly, in the later literature, the cone is called *alonee* and *etz'trovlee*, 'in the shape of the fir-cone': see Klatzkin, *Thesaurus Philosophicus*, I, pp. 48, 71.

and flat bottom,[1] be it quadrilateral, circular, or trilateral: you measure the area of the bottom, throw away two-thirds of the area and hold one-third, and multiply it into the height, and the result is the bulk of the body, from the top to the bottom.[2]

CHAPTER III

§ 1. There are five[3] kinds of quadrilaterals[4] as follows: (1) one with equal sides and angles; (2) one with unequal sides and equal angles;[5] (3) one with equal sides and unequal angles;[6] (4) one with ⟨equal and⟩ unequal sides and angles, [namely] with the two lengths equal for themselves and the two breadths [equal] for themselves; and (5) one with entirely unequal sides and angles.

1. *Sof* = 'end, bottom, lowest part, basis'; al-Khowarizmi has a word = 'lowest part, bottom'. This definition, 'having a sharp top and a flat bottom' certainly sounds more ancient and primitive than the definition given by Hero, p. 198, note 10, and Euclid, *Elements*, XI, def. 12, 18–20, Heath, III, pp. 268, 270. The conception and term of the mathematical point do not occur in the *Mishnat-ha-Middot*. In the Talmud we have the word *n'kudah* and *khod* for the dot and point in general, but not for the mathematical point.

2. The formula is *bh*/3. We know it on the authority of Archimedes that Eudoxus (*c.* 370 B.C.) was the first who proved this formula for the volume of the pyramid and cone; see Smith, I, p. 91, Cantor, I⁴, p. 241. The Hindus, however, as late as in the time of Aryabhata (*c.* 510) still had a wrong formula (*hb*/2) for the volume of the pyramid; see Smith, II, p. 293; Cantor, I⁴, p. 645. Maimonides, *Moreh N Vokheem*, I, 36, also speaks of the error of those that think the cone to be the half of the cylinder with the same base and height.

3. The same five kinds of quadrilaterals also in Euclid, I, def. 22; see Heath, *Euclid*, I, p. 188 seq. The Hindus have also five classes of quadrilaterals, but of different kind; see Cantor, I⁴, pp. 651, 727. Al-Khowarizmi, §9, follows the M.-ha-M.

4. The quadrilaterals are treated before the trilaterals like in I, 2, 3. The reason for that is that the triangle is regarded as half of the quadrangle; see I, 1; IV, 2. In Hero's works, too, the quadrangles precede the triangles; see Cantor, I⁴, pp. 389, 391, 395, 506.

5. The rectangle.

6. The rhombus.

CHAPTER IV

§ 1. There are three kinds of triangles as follows: The upright one,[1] the acute and the open[2] one. What is that an upright one? Its two short sides are multiplied each into itself [and added to each other]. And the long side which is the base is multiplied into itself. Then the last [square] is equal to the first one. For example: Six from this side and eight from that side, [and ten from the long side]; and the result of these [two each multiplied] into itself is hundred, and [the result] of that[3] [side] into itself is hundred. (If[4]) one wants to measure[5] [let him multiply one] of the short ones into the half of the other one, either 8 into 3 or 6 into 4, and the result is the area.

§ 2. And the angle standing between the short ones is the upright.[6] And this [triangle] is the half of a quadrilateral with unequal sides and equal angles.[7]

§ 5. If one wants to know the measure of the pillar in the equilateral [triangle], let him multiply one of the sides into itself [and the half of the other side into itself] and throw off the smaller [square] from the larger. The rest is the foundation[8] and what is found[9] is the pillar.

1. The name is taken from the solid upright standing triangle; see Gandz, *The Origin of Angle Geometry*, pp. 464 seq., 472 seq.

2. The obtuse. Above, III, 4, the oblique angles are called 'narrow' and 'broad'; see note 6 hereafter. 3. The hypotenuse, which is ten.

4. In round brackets is given the translation of the corrupt text, as presented by manuscript and editions. Subsequently, in square brackets the emendated text, as suggested by al-Khowarizmi's version, is given.

5. The triangle.

6. The right angle is called the 'upright angle' and is defined as standing between the short sides of the upright triangle. Similarly, in §§4 and 10, the acute and obtuse angles are described as occurring in the acute, respectively obtuse triangle; see *The Origin of Angle Geometry*, pp. 467 seq., 471 seq. The M.-ha-M., by its character, is nearer to Egyptian than to Greek mathematics. It has here (see §§ 4–9) the division of the trilaterals according to the sides, equilateral, isosceles or scalene, but not the grouping according to the angles. On the contrary the angles are grouped after the shape of the trilaterals, where they occur.

7. The rectangle. 8. The square.

9. By extracting the root. Read perhaps *v'ekro* 'and its root', like in the following paragraph.

§ 6. How does one calculate in numbers?[1] Ten upon ten is 100, the half of the other side, which is 5, multiplied into itself is 25. And he throws off the small from the large, [hence] there remains 75 and this is the foundation and its root is 8 and a rest.[2] If one

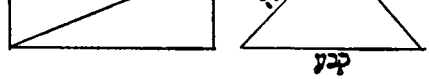


Fig. 10.1.

wants to measure let him multiply the root into the half of the bottom side[3] and the result of the computation is the area, which is 43 and a rest.

§ 7. (There[4]) are seven[5] other kinds of trilaterals. The equilateral, each side [multiplied] into itself. What is said from the one [side] is good for the other one. The one who understands the apple

1. Compare I, 6, p. 194, note 7.
2. Shirim is the usual term in the Talmud for a small balance in fractions.
3. The base.
4. In round brackets is given the translation of the text as presented by the manuscript. It seems to be corrupt. The whole paragraph seems to be a variant to §§ 5–6, which is indicated by the gloss s.a., or *sifrim akharim*, 'a reading of other books', that came into the text; see note 20 to the Hebrew text. This variant differs from § 6 in one very important point. In § 6 the root of 75 is given as 8 and a balance; it is multiplied into 5 and said to give 43 and a rest. But since the balance of $\sqrt{75}$ over 8 is not mentioned, we do not know how the result of 43 and a balance is obtained. Here, however, 5 is brought under the root, and we have $5\sqrt{75} = \sqrt{75{,}25} = \sqrt{1875} = 43$ . . . This is the method of al-Khowarizmi, § 10h; the transformation of $a\sqrt{b}$ into $\sqrt{a^2b}$ is taught in al-Khowarizmi's *Algebra*, pp. 19–20.
5. Steinschneider suggests that the word *shivah* was originally written for the letter zion indicating that it is the seventh Mishna. Schapira emendates into *yashbah*. If we maintain the reading *shvah*, however, this would give an interesting parallel to Hero. Hero, IV, pp. 38 and 92, speaks of six species of trilaterals, like Euclid; see Euclid, I, def. 20–21; Heath, *Euclid*, I, pp. 187–189. But on pp. 132, 150 and 269 ibid. Hero mentions seven kinds of trilaterals. They are: 1, the equilateral; 2–4, the right, acute, and obtuse isosceles; 5–7, the right, acute, and obtuse scalene. Similarly, Proclus has 7 kinds; see Heath, ibid., p. 188.

(?),[1] let him multiply one [side] into itself, which gives 100, and throw away the quarter, which is 25, there remain then 75. (If one wants to measure.) [There are other ways in an equilateral triangle. Each side is [measuring ten, f. e.] . . . If one wants to measure, let him multiply one [side] into itself, this gives 100, and throw away the quarter, which is 25, there remain then 75]; let him multiply 75 into 25, this is three-quarters[2] into one-quarter, this amounts to 1875; let him take the root of it, and this is the area and this is 43 and a rest. Thus you find here that the pillar[3] always falls upon the half of the base.

§ 8. You have, thus, found out the [area of the] equilateral and which is similar to it.[4] As for the calculation of the scalenes,[5] however, there is nothing more difficult than that in the calculation of the area. If one is searching for them, let him closely consider either the sides or the pillar and the base.[6]

§ 9. How[7] does one figure? For example: a triangle with un-

1. The meaning is dark. May be 'apple-numbers', or 'algebra'? Cf. Heath, *Greek Mathematics*, I, p. 15 says: 'Several of them (of the problems in the Greek anthology) are problems of dividing a number of apples or nuts among a certain number of persons'.

2. This does not mean the transformation of common into decimal fractions, as Schapira thinks. It rather gives the general formula for the altitude, as $\sqrt{\frac{3a^2}{4}}$ and for the area as $\sqrt{\frac{3a^2}{4}\cdot\frac{a^2}{4}}$. The Heronian formulae for the equilateral triangle are $\frac{\sqrt{3}}{4}a^2$ and $\frac{a^2}{3}\div\frac{a^2}{10}$, see Hero, III, p. 46 seq. and Cantor, I^4, pp. 376 and 399.

3. In the equilateral triangle.

4. The isosceles.

5. Literally *khalofim* = 'the different, unequal ones', i.e. the triangles with unequal sides.

6. In the following, only the first method, i.e. to compute the area by the three sides, is given. But the method of finding out the height and its footpoints is not given.

7. Here the so-called Heronian formula, $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$, is given, in words of course. The formula $\sqrt{s(s-a)(s-b)(s-c)}$ and the method of finding the perpendiculars of a triangle when the sides are given are ascribed, on the authority of al-Biruni (*c.* 1000), to Archimedes (*c.* 225 B.C.); see *Bibliotheca Mathematica*, XI, 3 (1910–11), pp. 37–40, 69; Heath, *Greek Mathematics*, II, p. 103; Smith, *History*, II, p. 287; Tropfke, IV^2, p. 9. The

equal sides and acute angles,[1] 15 from one side, 14 from the other side and 13 from the third side. Whoever wants to measure, let him seize the three of them together, they amount to 42: let him take the half [of it] and see how much greater it is than the first side, and let him multiply the half upon the difference, this is 21 into 6, which gives 126, and put it on the side; let him again, for the second time, take the half and see how much greater it is than the second side, and let him multiply the difference, which is 7, into the first 126, which amounts to 882, and put it on the side; let him again, for the third time, take the half and see how much greater it is than the third side, and let him multiply the difference, which is 8, into the last 882, and this amounts to 7056 and its root is 84, and this is the measure of the area.

CHAPTER V

§ 1. There are three kinds of the circular [figures] as follows: The suspended one,[2] the hill-like[3] and the flat one.[4] What is that the suspended one? That is one that stands on the root[5] of a circle from all the sides, [like] a sphere, or that the surface was like [that of] a water-melon which is circular all around, provided that its rounding is equal in length, breadth and depth.[6] How does one measure? Let him multiply the threads, the one into its own half, [and the result let him multiply into $3+\frac{1}{7}$], and the result is the area.[7] And double it, because they[8] are its walls.

Heronian formula became very popular. It is found in the Roman *agrimensores*; see Cantor, I[4], pp. 555, 590. Brahmagupta (*c.* 628) gives one and the same formula for triangle and quadrilateral, namely

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)};\ \text{in the triangle } d \text{ is } =0.$$

1. Here, exceptionally, the triangle is defined according to the angles; see p. 200, note 6.
2. The sphere.
3. The hemisphere.
4. The circle.
5. Here used perhaps in the meaning of 'side', or 'circumference'; compare III, 2, where the side of a square is called root, because it stands upon it. Hence the root of a circle would be the point upon which it stands.
6. A melon which is 'circular all around' and the rounding of which 'is equal in length, breadth and depth' is a spherical and not an elliptical one.
7. Of the hemisphere.
8. The two hemispheres.

§ 2. What is that the hill-like one? That is one that stands on[1] the half of the suspended one, like a hill or a vaulted tent, provided that it is even. If one wants to measure, let him multiply one of the threads [running] from one end to the other end into the half of the other one or into its own half, [and the result let him multiply into $3+\frac{1}{7}$] and the result is the area.

§ 3. What is that the flat one? That is one that is put down [flatly] on the ground like a circular field or a circular figure. If one wants to measure, let him multiply the thread into itself and throw away from it the one-seventh and the half of a seventh; the rest is the area, its roof. And if you want to know the circumference all around, multiply the thread into 3 and one-seventh[2] and it amounts to 22.[3] And if you want to compute the area, take the half of the circumference which is 11 and multiply it into the half of the thread[4] which is 3 and a half, and it amounts to 38 and a half. It is the same [result] according to the first [method] and the last one.

§ 4. Now[5] it is written:[6] 'And he made the molten sea of ten cubits from brim to brim, round in compass', and [none the less] its [circumference[7]] is thirty cubits, for it is written:[8] 'And a line of thirty cubits compassed it round about.' What is the meaning of the verse 'and a line of thirty cubits' a.s.f.? [Nehemiah[9] says]:

1. See p. 205, note 5.

2. On π as $3+\frac{1}{7}$ see the Introduction, notes 17, 18 and 20. On the history see still Tropfke, IV^2, pp. 196 seq., 205 seq.; Smith, 'The History and transcendence of π' in *Monographs on Modern Mathematics*, New York 1927, pp. 389 seq.; on the Babylonian value cf. Struve in QS, B 1, p. 85 seq., for the Egyptian value, see Struve, QS, A 1, pp. 177–9 and Neugebauer, QS, B 1, p. 429; for the Hindu value see B. Datta in *Journal and Proceedings, Asiatic Society of Bengal*, vol. 22, 1926, pp. 25–42.

3. The diameter is assumed to be seven, like in II, 3, in order to obtain a whole number for the circumference.

4. On the semidiameter.

5. Here seems to begin a new paragraph, § 4. Because both manuscripts, M. (according to Schapira) and B. agree in heading the next § with hay = 5. Here the text in MS. B. starts again.

6. I Kings, 7, 23; II Chronicles, 4, 2.

7. *Kumto* = 'height' is apparently corrupt.

8. Loc. cit.

9. This is the reading of B. נחמיה אוי.

Since the people of the world[1] say that the circumference of a circle contains three times and a seventh of the thread, take off from that one-seventh[2] for the thickness of the sea[3] on the two brims, then there remain 'thirty cubits compass it round about'. The seas, reservoirs and cisterns are equally [computed] according to this measure in length, breadth and depth. Thus you have learned the measure of the circular [figure].

§ 5. There are three kinds of the bow-figure, as follows: The even[4] one, the minor and the major one. What is that the even one? Each [arc] standing upon[5] the half of the circle, neither less nor more than it. The minor? Each [arc] being less than half the circle. And the major? Each [arc] being greater than half the circle. The rule is as follows: Each [arc] which has an arrow standing [upon the half of the chord[6] is known to be an even one. Each [arc] having an arrow standing] [upon] less than half of the chord is known to be a minor one. And each [arc] having an arrow greater than half of the chord is known to be a major.

§ 6. If one wants to measure the even one, let him multiply the entire chord into itself and throw away from that the seventh and half of the seventh.[7] From the rest let him throw away the half. The result is the area.

§ 7. Concerning the other ones[8] you have to know the measure of their circle.[9] How does one figure? Let him multiply the half

1. The gentiles, or the people devoted to secular things and secular learning, or *b nai aretz* = 'the landmeasurers'. In B. the phrase is missing; it is perhaps a corruption of *n'kham yah omer*?

2. Of the diameter.

3. Of the walls of the sea. According to this explanation the diameter of 10 included the walls, while the circumference excluded them. This is typical for the harmonizing, scholastic, exegesis of Mishna and Midrash.

4. The semicircle, because it divides the circle into two even parts.

5. i.e. equal to the half of the circle. 'When the arrow is equal to the half of the chord.'

6. The author apparently does not realize that the chord here is the diameter and that the diameter bisects the circle, or he does not care to say so.

7. That gives the area of the whole circle; see II, 3. Again, we would have expected the author to say that the chord is the diameter of the circle. The use of the formula shows that he knows it.

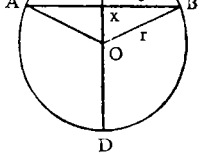
8. The minor and the major.

9. The diameter.

of the chord into itself, divide the product upon[1] the arrow and add the quotient to the arrow; the result is the thread of the circle.[2] Now let him take the half of this thread and multiply it into the half of the [arc] and let him put the result aside.[3] Now let him see [how the arc-figure is]. Was it a minor-arc figure, let him throw away its arrow from the half-thread[4] of the circle and multiply the rest into the half-chord and subtract it from the side;[5] the rest is the area. And if it was a major arc-figure, let him throw away the half-thread of the circle from its own arrow and multiply the rest into the half-chord and add it to the side;[6] the result is the area.

1. El = 'upon, through, by'.

2. See Euclid, *Elements*, III, 35; Heath, II, p. 71. $AE.BE = CE.DE$; hence $\frac{AE.BE}{CE} = DE$; cf. also Savasorda's geometry, § 33, p. 16, and § 106, p. 65, note. The formula is: $\frac{\left(\frac{c}{2}\right)^2}{a} + a = 2r$, if we call the chord c, the arrow a, and the radius r. This formula can also be found by the following equation: $r^2 = \left(\frac{c}{2}\right)^2 + x^2$; $x = r - a$; hence, $r^2 = \left(\frac{c}{2}\right)^2 + r^2 - 2ra + a^2$; $2ra = \left(\frac{c}{2}\right)^2 + a^2$; $2r = \frac{\left(\frac{c}{2}\right)^2}{a} + a$.



This formula occurs also in the works of Brahmagupta (*c.* 628) and Bhascara (*c.* 1150), quoted by Colebrooke, pp. 89 and 309, but without proof. Al-Karkhi, II, p. 24, and Savasorda, loc. cit., have the formula with the proof.

3. As the area of the sector, $br/2$ where b is the bow.

4. The radius.

5. From the product that was put aside as the area of the sector.

6. $\frac{br}{2} + \frac{(r-a)c}{2}$.

II

LIU HUI

(Third century)

LIU HUI (also, Lew Hway), a celebrated Chinese scholar and mathematician, lived in the Kingdom of Wei in North China during the period of the Three Kingdoms in the third century A.D. He is known to us through his commentaries on Chang Tsang's *Arithmetical Rules in Nine Sections*. The commentaries, written in 263, in the reign of Ch'en-lui-Wang of the Wei Kingdom, survive to this day. Antedating by a thousand years the development in China of the algebra of the celestial element, Liu Hui's work contains remarkable results achieved through his construction of rules which were effective as formulas for the solution of the problems proposed. In the commentaries, Liu Hui attempted the quadrature of the circle by the method of successively doubling the number of sides of an inscribed polygon. Starting with a regular hexagon inscribed in a circle of unit radius, he proceeded to the calculation of the length of a side of a polygon of 192 sides, and, in this work, developed the value 157:50 for the ratio (π) of the circumference of the circle to its diameter. Liu Hui noted, moreover, that this ratio was somewhat smaller than the actual value.

Liu Hui included some problems of his own in his commentaries on the ancient classic. A supplement consisting of problems in practical trigonometry with their solutions was originally appended to the last book of the old 'Nine Sections'. This group of problems appeared later as a separate work under the title *Chung-ch'a*, which refers to the use of stiles of different lengths for taking observations and to the repeated applications of the right triangle relations in the resulting figure. During the T'ang (618–906) the title *Chung-ch'a* was changed to *Hai-tao Suan-ching* (sometimes also, *Hai-tao Suan-shu*), 'Sea-Island Arithmetic Classic', the new title having been taken from the first problem of the *Chung-ch'a* dealing with the measure of an island lying at a distance from the observer. Liu Hui's treatise was one of the nine Chinese works specified as approved mathematical studies in the Japanese university established in 701 by the Emperor Monbu. We are indebted to many

scholars for the survival of the work. An edition of it was issued (1775) by the illustrious scholar Tai Cheng, imperial librarian and mathematician, who drew the various parts of the work from the great encyclopaedia *Yung-lo Tai-tien* of the Yung-lo period (1403–24).

SEA-ISLAND ARITHMETIC CLASSIC

From The Development of Mathematics in China and Japan
by Yoshio Mikami

The first problem in the 'Sea-island Arithmetical Classic' is this:

"There is a sea island that is to be measured. Two rods that are both 30 feet (a) high are erected at the distance of 1,000 paces (b) from each other, making the rod in the rear to come in the same straight line with the island and the other rod. When a man walks 123 paces (c) back from the nearer rod, the top of the island is just visible through the end of that rod, if he tries to see with his eye brought on the ground. The summit of the island peak is also seen co-stra'ght with the end of the hind rod, when seen bringing the eye in contact with the ground from a point 127 paces (d) back from that rod. It is required to know the height and the distance of the island.'

The rule given for the solution of this problem runs:

'Rule. With the height of the rods multiply the distance of the rods, and take (the product) as the dividend; and carry the division taking the difference (of the distances walked back from the rods) as the divisor; add to the quotient the rod's height, when we obtain the height of the island.

'To find the distance of the island from the nearer rod, multiply with the distance walked back from that rod the distance between the two rods, and divide the product by the difference of the distances walked back from the rods, when the quotient gives the distance required.'

This rule is equivalent to the formulae,

$$\text{(height)} = a + \frac{ab}{d-c}; \qquad \text{(distance)} = \frac{cb}{d-c}.$$

It is not known how Liu Hui got at these expressions, but it

appears not improbable that they were deduced from the two proportions,

$$\frac{x}{x+c}=\frac{a}{c}, \quad \text{and} \quad \frac{x+b}{x+b+d}=\frac{a}{d},$$

or what are equivalent to them.

The term *chung-ch'a* seems to indicate an agreement with our view. The term was originally intended to mean double or repeated applications of the consideration as met with in the chapter on the right triangle in the 'Nine Sections', i.e. the double application of proportions.

The second of Liu Hui's problems is as follows:

'There grows on a hill a pine-tree, whose height is not known. Two rods, each 20 feet high (*a*), are so erected in the plain below at the mutual distance of 50 paces (*b*), that they come in a straight line with the tree. The top of the tree and the end of the front rod make a straight line with a point on the ground 7 paces and 4 feet (*c*) behind the rod, and at this point the base of the tree measures 2·8 feet (*e*) on the top of the rod. The top of the tree again makes a straight line with the end of the hind rod and a point on the ground 8 paces and 5 feet (*d*) behind the rod. It is required to know the height of the pine-tree and the distance of the hill from the (front) rod.'

The rule that is intended to solve this problem leads to the formulae,

$$\text{(height)}=\frac{eb}{d-c}+e; \qquad \text{(distance)}=\frac{bc}{d-c}.$$

The third problem is:

'One sees to the south a square town of unknown size. He erects two rods 60 feet (*a*) apart from east to west, which are joined with a string at the eye's height, and let the eastern rod be co-straight with the S.E. corner and the N.E. corner of the town. He gets 5 paces (*b*) to the north of the eastern rod, and observes the N.W. corner of the town through a point of the string at 22·6½ feet (*c*) from the east end. Again getting back to the north 13 paces and 2 feet (*d*) from the rod, the observer sees the N.W. corner of the town just co-straight with the western rod. What are then one side of the town and the distance of the town from the rod?'

This problem is the same as asking for the lengths (see the figure) $CB=x$ and $BD=y$ in terms of $DE=a$, $DF=b$, $DG=c$, $DA=d$. [In the figure B is a right angle and DE is drawn parallel to BC.]

The formulae in the answer are given in forms equivalent to

$$x=\frac{(d-b)c}{\frac{cd}{a}-b},$$

$$y=\frac{\left(d-\frac{cd}{a}\right)b}{\frac{cd}{a}-b}.$$

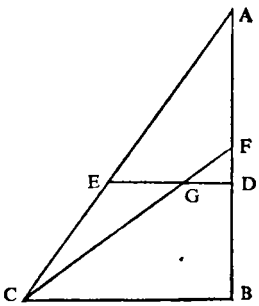


Fig. 11.1.

Liu Hui gives in his *Arithmetical Classic* six further similar problems, together with the rules for their solutions.

Liu Hui's studies of these complicated problems appear to tell that he was acquainted with some way of algebraical manipulations. When we moreover reflect on the contents of some parts of the 'Nine Sections', we cannot but ascribe the birth of algebra to Chinese mathematicians in a comparatively early part of their history, much earlier than the time of the appearance of Ch'in Chiu-shao and Li Yeh in the thirteenth century, when, as is generally believed, the Chinese algebra was born. We put therefore a stress on Liu Hui's considerations of these complicated problems, that had taken place in the third century, a whole millennium previous to the dawning of the algebra of the celestial element.

12

SUN-TSU

(*c.* Third Century)

SUN-TSU, an ancient Chinese mathematician of considerable power, was the author of a mathematical work, the *Sun-Tsu Suan-ching* 'Arithmetic Classic of Sun-Tsu', a treatise referred to as authoritative by almost every subsequent Chinese writer on mathematics. It was the first of the ancient Chinese works to consider the solution of indeterminate problems and it is the source of the famous 'Chinese remainder problem', where a number is sought such that if it is divided by certain divisors, then corresponding given remainders are obtained. The date at which Sun-Tsu lived is in some doubt. He was at one time identified with the military writer Sun-Wu-Tsu (or Sun-Tsu) of the fifth or sixth century B.C., whose treatise on the Art of War was the subject of many commentaries over the centuries. This identification has been definitely rejected on philological grounds, and Sun-Tsu, the mathematician, has been placed as belonging to the third century, at the end of the Han or the beginning of the Wei Dynasty. There are some authorities who believe that he may have lived as early as the first century A.D.

The *Sun-Tsu Suan-ching* is an elaborately constructed treatise in three books. It was based, in all likelihood, upon the earlier 'Arithmetical Rules in Nine Sections', but the work shows substantial progress over the older treatise in the depth and extent of the author's understanding of mathematical techniques. Sun-Tsu's explanation of the decimal number system employed in this arithmetic involved the knowledge and use of calculating rods. His authority in this area was unchallenged for many centuries. In later times, when a new practice arose for the arrangement of the calculating pieces representing the even and odd places in a number, his name was attached to the 'old method', being referred to as the *Son-shi-Reppu-ho*, 'the method of arrangement of Sun-Tsu'. Beginning with an exposition of metrological units and elementary operations in the *Sun-Tsu Suan-ching*, and proceeding to the method of extracting a square root, Sun-Tsu included all the topics that were taught in the 'Nine Sections'. The introduction of the solution of

indeterminate problems in this work was an important innovation. Sun-Tsu's rule for the solution of an indeterminate problem, *Ta-Yen* (or *T'ai-yen*), 'Great Extension' or 'Great Generalization', was given under the heading *Wuh-puh-chi-soo*, 'Unknown Numerical Quantities'. Stated in four lines of rhyme, the rule was followed by the problem itself and a brief solution. The *Sun-Tsu Suan-ching* became widely known in the West through Alexander Wylie's famous and highly informative article 'Jottings on the Science of Chinese Arithmetic'.

THE SUN-TSU SUAN-CHING

From The Development of Mathematics in China and Japan
by Yoshio Mikami

Sun-Tsu ascribes, in the preface of his work, the origin of all things, material as well as intellectual, to the science of numbers or of calculation.

At the beginning of Sun-Tsu's treatise he gives the names of great and small numbers and of the measures and weights.

We here quote from Sun-Tsu's text:

'The measure originates with *hu*. What is one *hu*? The silk-worm vomits its string, which measures (in diameter) one *hu*. Ten *hu* make one *mi*; ten *mi* one *hao*; ten *hao* one *li*; ten *li* one *fen*; ten *fen* one *t'sun*; ten *t'sun* one *ch'o* or one foot; ten feet one *chang*; ten *chang* one *yin*. And 50 feet make one *tuan*, 40 feet one *p'i*, 6 feet one *p'ou* or one pace, 240 paces one *mou*, and 300 paces one *li* or one Chinese mile.

'The weight originates with one grain of panicum maliaceum; ten of such grains make one *t'sen*; ten *t'sen* one *shu*; 24 *shu* one *liang*; 16 *liang* one *chin*; 30 *chin* one *chun*; and 4 *chun* one *shih*.

'The solid measure originates in a *su* or a grain of millet; six such grains make one *kuei*; ten *kuei* one *t'so*; ten *t'so* one *ch'ao*; ten *ch'ao* one *shao*; ten *shao* one *ho*; ten *ho* one *shong*; ten *shong* one *tou*; ten *tou* one *hu*; which contains 60,000,000 grains of millet.'

Sun-Tsu takes $\pi = 3$, and the ratio of the side and the diagonal of a square to be 5 to 7.

The weights of gold, silver, jewel, copper, lead, iron and stone

are given to be one *chin*, 14, 12, $7\frac{1}{2}$, $9\frac{1}{2}$, 6 and 3 *liangs*, respectively, for one *t'sun* cube.

Sun-Tsu does not explain expressly the use of the calculating pieces, but what he records in the first book of his treatise seems to indicate clearly the recourse being taken to that sort of abacus.

The calculating pieces are made of small bamboo or wooden pieces of two colours, red and black, representing the positive and the negative numbers. The pieces that were used in the later growth of mathematics were some $1\frac{1}{2}$ inches long and a sixth part of their length broad. But in old times much longer pieces made of bamboo were in use, as is recorded in the *Han Shu*, written by Pan Ku at the end of the first Christian century. Although we are not certain how long the abacus had been used in China, yet we are sure that it was not first brought into use during the ages of the Han Dynasty; it had most likely been employed from time immemorial in China. A passage in the illustrious book, *Tso Ch'uan*, serves to confirm this view; and the event told there relates to an occurrence in the year 542 B.C.

Sun-Tsu says in the first book of this treatise:

'In making calculations we must first know positions of numbers. Unity is vertical and ten horizontal; the hundred stands while the thousand lies; and the thousand and the ten look equally, and so also the ten thousand and the hundred.'

This stage in Sun-Tsu is nothing but a description of the arrangement of the calculating pieces. These pieces are usually arranged in the following manner:

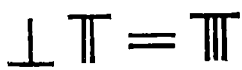
The first five numbers are represented by 1, 2, 3, 4 and 5 of these pieces, that are arranged vertically, while in the arrangement of the numbers from 6 to 9 one piece taken in the horizontal position over other vertical pieces is taken for 5. Thus:

This way of arrangement is referred to in Sun-Tsu's work in the paragraph that follows that which we have just quoted; and it is also in sound accordance with what we can conjecture from the

obscure passage in the *Tso Ch'uan* mentioned above. Moreover, this was the very way that was followed in subsequent ages.

The figures in the ten's column are arranged with the calculating pieces in the following manner:

For hundreds and higher numbers these arrangements are repeated alternately, but nothing new comes in. Thus the number 6728 will be arranged as annexed.



The explanation of the extraction of the square-root found in the second book of Sun-Tsu:

'Arrange the area 234567 paces square and make it the *shih*.

'Next borrow a unit calculator, and make it the *hsia-fa*'.[1]) Thus:

2	3	4	5	6	7	*shih*
					1	*hsia-fa*

'Advance it over every one column, thus dividing the *shih* in groups, until it reaches the hundred's place.' Thus:

2	3	4	5	6	7	*shih*
	1					*hsia-fa*

'Arrange the *shang* or root 400 above the *shih*.

'Subjoin 40,000[2]) under the *shih* and above the *hsia-fa*, and make it the *fang-fa*.

1. The word *hsia* means below or under.
2. This is the product of the root and the *hsia-fa*.

'With it multiply the root 400 and subtract the product from the *shih.*' Thus:

	4		0		0	root
	7	4	5	6	7	*shih*
	4	0	0	0	0	*fang-fa*
	1					*hsia-fa*

'The subtraction being done, double the *fang-fa*, and draw it back one column, and the *hsia-fa* two columns.' Thus:

4		0		0	root
7	4	5	6	7	*shih*
	8	0	0	0	*fang-fa*
		1			*hsia-fa*

'Again arrange 80 for the root next to the previous digit.

'Subjoin 800 under the *fang-fa* and above the *hsia-fa*; nominate it the *lien-fa.*

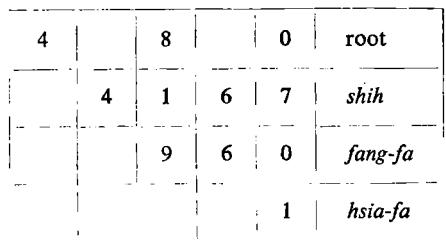
'Multiply the *fang-fa* and the *lien-fa* by the root 80, and subtract the products from the *shih.*' Thus:

4		8		0	root
	4	1	6	7	*shih*
	8	0			*fang-fa*
		8			*lien-fa*
		1			*hsia-fa*

'When these subtractions are done, double the *lien-fa*, and add to the *fang-fa*.

'Draw back the *fang-fa* one column, and the *hsia-fa* two columns.'

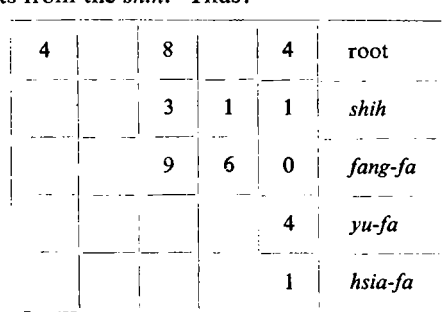
Here the *lien-fa* is understood to be got rid of. And thus the arrangement becomes:



'Again, arrange 4 in the next column of the previous root.

'Subjoin 4 under the *fang-fa* and above the *hsia-fa*, and name it the *yu-fa*[1]).

'Multiply the *fang*, *lien*, and *yu*[2]) by the root 4, and subtract the products from the *shih*.' Thus:



'This being over, double the *yu-fa* and add it to the *fang-fa*.

'Thus we get 484 for the root in the above and 968 for the *hsia-fa*[3]), the remainder being 311.'

1. This is the same as the previous *lien-fa*, only that it is differently named.

2. This passage may seem somewhat strange, but it means that the *fang-fa* and the *yu-fa* are to be multiplied by 4 in the root.

3. This is not in the same meaning as previously.

'One side (of the square) is therefore 484-311/968 paces.'

We here state one or two problems from Sun-Tsu's *Arithmetical Classic*, Book 3.

'A woman was washing dishes in a river, when an official whose business was overseeing the waters demanded of her: "Why are there so many dishes here?" – "Because a feasting was entertained in the house", the woman replied. Thereupon the official inquired of the number of the guests. "I don't know", the woman said, "how many guests had been there; but every two used a dish for rice between them; every three a dish for broth; every four a dish for meat; and there were sixty-five dishes in all."

'Rule. Arrange the 65 dishes, and multiply by 12, when we get 780. Divide it by 13, and thus we obtain the answer.'

'There are certain things whose number is unknown. Repeatedly divided by 3, the remainder is 2; by 5 the remainder is 3; and by 7 the remainder is 2. What will be the number?'

This problem reduces to the indeterminate equations

$$3m_1+2=5m_2+3=7m_3+2,$$

of which the solution is

$$m_1=5\times 7r+7, \quad m_2=21r+4, \quad m_3=3\times 5r+3,$$

for $r=0, 1, 2, 3, \ldots$

Sun-Tsu tries to solve this problem thus:

'The remainder divided by 3 is 2, and so take 140. The remainder divided by 5 is 3, and so take 63. The remainder divided by 7 is 2, and so take 30. Adding these together we get 233 Therefrom subtract 210, and we obtain the answer.'

Sun-Tsu thus satisfied himself with a single solution of the problem.

Further adds Sun-Tsu: 'In general, take 70, when the remainder of the repeated divisions by 3 is 1; take 21, when the remainder of the repeated divisions by 5 is 1; and take 15, when the remainder of the repeated divisions by 7 is 1. When the sum of these numbers is above 106, subtract 105, before we get the answer.'

All the subjects treated by Sun-Tsu were those that had been

given in the *Arithmetic in Nine Sections*, but the problem of indeterminate analysis first appeared in the above in Sun-Tsu.

We find in Sun-Tsu a problem of finding the least common multiple. It is this:

'There are three sisters, of whom the eldest comes home once in every 5 days, the middle in every 4 days, and the youngest in every 3 days. In how many days will all the three meet together?'

The answer is 60 days.

To solve it Sun-Tsu gives the rule in these words:

'Arrange the 5 days of the eldest sister, the 4 days of the middle and the 3 days of the youngest in the right column. Opposite to each of these arrange unity on the left column. Then cross multiply (the numbers on the right), when we get the numbers of their returning. Thus the eldest comes 12 times, the middle 15 times and the youngest 20 times. If we multiply these numbers by the numbers of days in which the sisters once come back, we shall obtain the desired answer.'

The last problem in the *Arithmetical Classic* of Sun-Tsu is one that relates to fortune-telling. Thus:

'A pregnant woman, who is 29 years of age, is expected to give birth to a child in the 9th month of the year. Which should be her child, a son or a daughter?'

Sun-Tsu gives to this problem the rule:

'Take 49; add the month of her child-bearing; subtract her age. From what now remains, subtract the heaven 1, subtract the earth 2, subtract the man 3, subtract the four seasons 4, subtract the five elements 5, subtract the six laws 6, subtract the seven stars 7, subtract the eight winds 8, subtract the nine provinces 9. If then the remainder be odd, the child shall be a son; and if even, a daughter.'

Yuan Yuan desires in his *Biographical Collections* to ascribe this one problem to a subsequent addition, not to the original text of Sun-Tsu, since it appears to him too unbecoming for so able a writer as Sun-Tsu.

13

ĀRYABHATA, THE ELDER

(b. A.D. 476)

ĀRYABHATA, the earliest Indian author on scientific astronomy whose work is extant, was the author of the *Āryabhatīya*, the first Indian astronomical text to contain a section devoted entirely to basic mathematics. The work, written in extremely concise, frequently poetical style, presents a series of rules apparently presupposing oral instructions to which the rules were a supplement. No detailed instruction is given, There may perhaps have been a more detailed work not yet rediscovered accompanying the rules.

Āryabhata states in his *Āryabhatīya* that he was born in A.D. 476 and that he was twenty-three years of age when he wrote this work in the city of Kusumapura ('City of Flowers'). By this statement, the scene of his major scientific activity was a city just above the Ganges near the present Patna which the ancient Buddhists called Pataliputra. These are the cities which, on various grounds, have been mentioned by historians as the place of his birth.

The text of the *Āryabhatīya* indicates that Āryabhata made astronomical observations, that he had a device for measuring time intervals, that he systematized previous astronomical achievements, discarding such as were in error, and that he possessed the technical mathematical knowledge needed for his astronomical calculations. The thirty-three verses specifically on mathematics were brief rules for computational procedures including the extraction of square and cube roots, the solution of quadratic equations, the sum of the powers of the first n natural numbers and the sine or semi-chord.

Numerous later writers explained and enlarged the *Āryabhatīya*, although at times there was no lack of criticism of some particulars of his astronomical beliefs which ran counter to tradition. Brahmagupta especially, made frequent adverse mention of Āryabhata. Indeed, in his *Brahmasphutasidd'hanta* (A.D. 628) much of the eleventh chapter is given to a bitter denunciation of Āryabhata's innovations. These criticisms themselves became the subject of long and protracted controversy, in

the course of which the *Āryabhatīya* was again closely studied. Under the subsequent intensive scrutiny, the words of Āryabhata gained in respect and honour, and as the centuries passed they continued to influence and inspire succeeding generations of scholars.

THE ĀRYABHATĪYA OF ĀRYABHATA[1]

translated with notes by Walter Eugene Clark

CHAPTER I. *Dasagitika* or 'The Ten Gīti Stanzas'

A. Having paid reverence to Brahman, who is one (in causality, as the creator of the universe, but) many (in his manifestations), the true deity, the Supreme Spirit, Āryabhata sets forth three things: mathematics [*ganita*], the reckoning of time [*kālakriyā*] and the sphere [*gola*].

B. Beginning with *ka* the *varga* letters (are to be used) in the *varga* places, and the *avarga* letters (are to be used) in the *avarga* places. *Ya* is equal to the sum of *na* and *ma*. The nine vowels (are to be used) in two nines of places *varga* and *avarga*. *Navanty-avarge va*.

The words *varga* and *avarga* seem to refer to the Indian method of extracting the square root . . .

The *varga* or 'square' places are the first, third, fifth, etc., counting from the right. The *avarga* or 'non-square' places are the second, fourth, sixth, etc., counting from the right. The words *varga* and *avarga* seem to be used in this sense in II, 4.

CHAPTER II. *Ganitapada* or 'Mathematics'

1. Having paid reverence to Brahman, the Earth, the Moon, Mercury, Venus, the Sun, Mars, Jupiter, Saturn, and the asterisms, Āryabhata sets forth here [in this work] the science which is honoured at Kusumapura.[2]

1. ED. NOTE: Those paragraphs opening with letters A, B, and numbers 1 to 26 are translations of the *Āryabhatīya*; the material following those paragraphs is commentary by Walter Eugene Clark, the translator.

2. Translated by Fleet, *JRAS*, 1911, p. 110.

The translation 'here at Kusumapura the revered science' is possible. At any rate, Āryabhata states the school to which he belongs. Kusumapura may or may not have been the place of his birth.

2. The numbers *eka* (one), *dasa* (ten), *sata* (hundred), *sahasra* (thousand), *ayuta* (ten thousand), *niyuta* (hundred thousand), *prayuta* (million), *koti* (ten million), *arbuda* (hundred million), and *vrnda* (thousand million) are from place to place each ten times the preceding.[1]

The names for classes of numbers are given only to ten places, although 1, B describes a notation which reaches at least to the eighteenth place. The highest number actually used by Āryabhata himself runs to ten places.

3. A square, the area of a square, and the product of two equal quantities are called *varga*. The product of three equal quantities, and a solid which has twelve edges are called *ghana*.

4. One should always divide the *avarga* by twice the (square) root of the (preceding) *varga*. After subtracting the square (of the quotient) from the *varga* the quotient will be the square root to the next place.

Counting from right to left, the odd places are called *varga* and the even places are called *avarga*. According to Paramesvara, the nearest square root to the number in the last odd place on the left is set down in a place apart, and after this are set down the successive quotients of the division performed. The number subtracted is the square of that figure in the root represented by the quotient of the preceding division. The divisor is the square of that part of the root which has already been found. If the last subtraction leaves no remainder the square root is exact. 'Always' indicates that if the divisor is larger than the number to be divided a zero is to be placed in the line (or a blank space left there).

*

The words of Āryabhata exactly fit the method employed in later Indian mathematics. . . .

1. See *JRAS*, 1911, p. 116.

In the following example the sign ° indicates the *varga* places, and the sign — indicates the *avarga* places.

	°–°–° 15129 (root=1)
Square of the root	1
Twice the root	2)05(2=quotient (or next digit of root)
(2×1)	4
	11
Square of the quotient	4
Twice the root	24)72(3=quotient (or next digit of root)
(2×12)	72
	09
Square of the quotient	9
	0

Square root is 123

5. One should divide the second *aghana* by three times the square of the (cube) root of the (preceding) *ghana*. The square (of the quotient) multiplied by three times the *purva* (that part of the cube root already found) is to be subtracted from the first *aghana*, and the cube (of the above division) is to be subtracted from the *ghana*.

Counting from right to left, the first, fourth, etc., places are named *ghana* (cubic); the second, fifth, etc., places are called the first *aghana* (non-cubic) places; and the third, sixth, etc., places are called the second *aghana* (non-cubic) places. The nearest cube root to the number in (or up to and including) the last *ghana* place on the left is the first figure of the cube root. After it are placed the quotients of the successive divisions. If the last subtraction leaves no remainder the cube root is exact.

In the following example the sign ° indicates the *ghana* places and the sign — indicates the *aghana* places.

	°—°—° 1860867 (root=1)
Cube of root	1
Three times square of root (3×1^2)	3)08(2=quotient (or next
	6 digit of root)
	26

Square of quotient multiplied by three times the *purva* ($2^2 \times 3 \times 1$)	12	
	140	
Cube of quotient	8	
Three times square of root (3×12^2)	432)1328(3	= quotient (or next digit of root)
	1296	
	326	
Square of quotient multiplied by three times the *purva* ($3^2 \times 3 \times 12$)	324	
	27	
Cube of quotient	27	
	0	

Cube root is 123

9. . . . The chord of the sixth part of the circumference is equal to the radius.

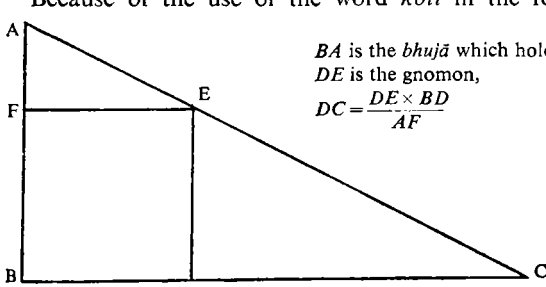
10. Add 4 to 100, multiply by 8, and add 62,000. The result is approximately the circumference of a circle of which the diameter is 20,000.

The circumference is 62,832. The diameter is 20,000.

By this rule the relation of circumference to diameter is 3·1416.

15. Multiply the length of the gnomon by the distance between the gnomon and the *bhujā* and divide by the difference between the length of the gnomon and the length of the *bhujā*. The quotient will be the length of the shadow measured from the base of the gnomon.

Because of the use of the word *koti* in the following rule



BA is the *bhujā* which holds the light,
DE is the gnomon,
$DC = \frac{DE \times BD}{AF}$

Fig. 13.1.

Rodet is inclined to think that the gnomon and the *bhujā* were not perpendicular but projected horizontally from a wall. *Bhujā* denotes any side of a triangle, but *koti* usually refers to an upright. It is possible, however, for *koti* to denote any perpendicular to the *bhujā* whether horizontal or upright.

17a. The square of the *bhujā* plus the square of the *koti* is the square of the *karna*.

17b. In a circle the product of two *saras* is the square of the half-chord of the two arcs.

The *bhujā* and *koti* are the sides of a right-angled triangle. The *karna* is the hypotenuse.

The *saras* or 'arrows' are the segments of a diameter which bisects any chord.

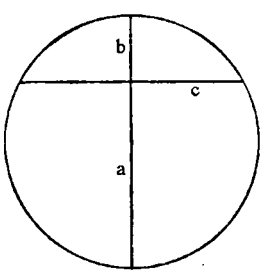


Fig. 13.2.

$a \times b = c^2$
where c is the half-chord.

19a. The desired number of terms minus one, halved, plus the number of terms which precedes, multiplied by the common difference between the terms, plus the first term, is the middle term. This multiplied by the number of terms desired is the sum of the desired number of terms.

19b. Or the sum of the first and last terms is multiplied by half the number of terms.

This rule tells how to find the sum of any desired number of terms taken anywhere within an arithmetical progression. Let n be the number of terms extending from the $(p+1)$th to the $(p+n)$th terms in an arithmetical progression, let d be the common difference between the terms, let a be the first term of the progression, and l the last term.

The second part of the rule applies only to the sum of the whole progression beginning with the first term.

$$S=n\left[a+\left(\frac{n-1}{2}+p\right)d\right],$$

$$S=\frac{(a+l)n}{2}.$$

20. Multiply the sum of the progression by eight times the common difference, add the square of the difference between twice the first term and the common difference, take the square root of this, subtract twice the first term, divide by the common difference, add one, divide by two. The result will be the number of terms.

$$n=\frac{1}{2}\left[\frac{\sqrt{8dS+(d-2a)^2}-2a}{d}+1\right].$$

As Rodet says, the development of this formula from the one in the preceding rule seems to indicate knowledge of the solution of quadratic equations in the form $ax^2+bx+c=0$.

22. The sixth part of the product of three quantities consisting of the number of terms, the number of terms plus one, and twice the number of terms plus one is the sum of the squares. The square of the sum of the (original) series is the sum of the cubes.

From the series 1 2 3 4, etc., form the series 1 4 9 16, etc., and 1 8 27 64, etc., consisting of the squares and cubes of the terms of the first series. The rule tells how to find the sums of the second and third series.

The rule for finding the sum of the first series was given above in stanza 19.

The sum of the squares is

$$\frac{n(n+1)(2n+1)}{6}.$$

23. One should subtract the sum of the squares of two factors from the square of their sum. Half the result is the product of the two factors.

$$ab=\frac{(a+b)^2-(a^2+b^2)}{2}.$$

24. Multiply the product (of two factors) by the square of two

(4), add the square of the difference between the two factors, take the square root, add and subtract the difference between the two factors, and divide the result by two. The results will be the two factors.

$$\frac{\sqrt{4ab+(a-b)^2}\pm(a-b)}{2} \text{ will give } a \text{ and } b.$$

26. In the rule of three multiply the fruit by the desire and divide by the measure. The result will be the fruit of the desire.

The rule of three corresponds to proportion.

In the proportion a is to b as c is to x the measure is a, the fruit is b, the desire is c, and the fruit of the desire is x.

$$x=\frac{bc}{a}.$$

14

BRAHMAGUPTA

(b. A.D. 598)

BRAHMAGUPTA, the author of the celebrated astronomical work, the *Brahma-sidd'hanta* (astronomical system of Brahma), sometimes called the *Brahma-sphuta-sidd'hanta* (correct astronomical system of Brahma), was the chief astronomer of the observatory at Ujjain in the early part of the seventh century. In a period of the history of civilization, when the world was at the nadir of its intellectual development, this illustrious Indian mathematician wrote a work on astronomy and mathematics which was to be studied and repeated word by word for centuries to come. Later Indian writers painstakingly explained each verse of Brahmagupta's text, adding illustrative examples for clarification. In the twelfth century, when Bhascara Acharya was head of the observatory at Ujjain (*c.* 1150), Brahmagupta's text and the commentaries on it were fully available to him. The *Brahma-sidd'hanta*, written when Brahmagupta was thirty years old, at a date which has been found to be equivalent to A.D. 628, had been preserved through the devotion of copyists and scholars. Through sustained interest and effort, it has survived to our time in almost complete form. Of the twenty-one chapters into which the treatise was originally divided, the first ten were on astronomy. The eleventh chapter was given over to a forceful rejection or correction of views which Brahmagupta regarded as erroneous. Chapter twelve was on arithmetic and mensuration. The next five chapters, in which there are now some serious gaps, were again on astronomy. The subject-matter of the eighteenth chapter was mathematical. The next two chapters are missing. The last chapter, the twenty-first, was on spherics. While basing his work on previous treatises, Brahmagupta justified his correction of older writers on the grounds of observation and tradition. 'The computation of the planets taught by Brahma,' he wrote, 'which had become imperfect by great length of time is propounded correct by Brahmagupta, son of Jishnu.'[1] The com-

1. H. T. Colebrooke, *Algebra with Arithmetic and Mensuration*, translated from the Sanskrit of Brahmagupta and Bhascara, p. xxx.

mentary available to us was written about A.D. 860 by the great scholiast of the *Brahma-sphuta-sidd'hanta*, Chaturveda-Prit'hudaca-Swami, frequently mentioned by Bhascara in his *Sidd'hanta-siromani* as one of the authorities upon whom he depended.

Most of the problems proposed by Brahmagupta in Chapter 12, *Ganitad'hyaya* (arithmetic and mensuration) and Chapter 18, *Cuttacad'hyaya* (algebra) were related to astronomy. Rather fanciful problem situations were introduced later, in the commentaries. Brahmagupta himself departed at times from the strictly practical problem. 'These questions are stated merely for gratification,' he stated toward the end of his eighteenth chapter. 'The proficient may devise a thousand others; or may resolve, by the rules taught, problems proposed by others.'[1] Among the mathematical topics taught by Brahmagupta were rules for elementary operations with positive and negative numbers, progressions, rational triangles and quadrilaterals, linear functions made squares, and problems solved by indeterminate equations of the first and second degrees. His work, especially in number theory, was followed and developed by a long list of famous mathematicians.

BRAHMA-SPHUTA-SIDD'HANTA

translated from the Sanscrit by Henry Thomas Colebrooke

CHAPTER XII: *Arithmetic*

Section I: Rule of Proportion

1. He, who distinctly and severally knows addition and the rest of the twenty logistics, and the eight determinations including measurement by shadow,[2] is a mathematician.[3]

1. H. T. Colebrooke, *Algebra with Arithmetic and Mensuration*, p. 377.

2. Addition, subtraction, multiplication, division, square, square-root, cube, cube-root, five [should be, six] rules of reduction of fractions, rule of three terms [direct and inverse], of five terms, seven terms, nine terms, eleven terms, and barter, are twenty (*paricarman*) arithmetical operations. Mixture, progression, plane figure, excavation, stack, saw, mound, and shadow, are eight determinations (*vyavahara*). – Ch.

3. *Ganaca*, a calculator; a proficient competent to the study of the sphere. – Ch.

10. In the rule of three, argument, fruit and requisition [are names of the terms]: the first and last terms must be similar.[1] Requisition, multiplied by the fruit, and divided by the argument, is the produce.[2]

Section III: Progression

17. The period less one, multiplied by the common difference, being added to the first term, is the amount of the last. Half the sum of the last and first terms is the mean amount: which, multiplied by the period, is the sum of the whole.[3]

1. The middle term is dissimilar. – Ch.

2. The rule concerns integers. If there be fractions among the terms, reduce all to the same denominator. – Ch.

Example: A person gives away a hundred and eight cows in three days; how many kine does he bestow in a year and a month?

Statement: Days 3. Cows 108. Days 390.

Answer: 14,040.

Example: A white ant advances eight barley corns less than one-fifth part of that amount in a day; and returns the twentieth part of a finger in three days. In what space of time will one, whose progress is governed by these rates of advancing and receding, proceed one hundred *yojanas*?

Example: One bestows an unit on holy men, in the third part of a day; another gives the same alms in half a day; and a third distributes three in five days. In what time, persevering in those rates, will they have given a hundred?

Statement: 1 1 3 And, the rule being observed, 3/1 2/1 3/5.
1 3 1/2 5/1

Reducing these to a common denominator, and summing them, the result is 28/5; the total amount, which all bestow in alms in a day. Then by the rule of three, if so many fifths of an unit be given in one day, in how many will a hundred units be given?

Statement: 28/5 | 1/1 | 100/1. Answer: 17 6/7. – Ch.

3. To find the contents of a pile in the form of half the *meru-yantra* [or spindle]. – Ch.

Example: A stack of bricks is seen, containing five layers, having two bricks at the top, and increasing by three in each layer: tell the whole number of bricks.

Statement: Init. 2; Diff. 3; Per. 5. Answer: 40.

Example: The king bestowed gold continually on venerable priests, during three days and a ninth part, giving one and a half [*bharas*] with a daily increase of a quarter: what were the mean and last terms, and the total?

Statement: Init. 1-1/2; Diff. 1/4; Per. 3-1/9.

Period 28/9, less one, is 19/9; multiplied by the difference, it is 19/36;

18. Add the square of the difference between twice the initial term and the common increase, to the product of the sum of the progression by eight times the increase: the square-root, less the foregoing remainder divided by twice the common increase, is the period.[1]

Section IV: Plane Figure.[2] *Triangle and Quadrilateral*

21. The product of half the sides and countersides[3] is the gross area of a triangle and tetragon.[4] Half the sum of the sides set

and added to the first term, becomes 73/36. This is the last term. Added to the first term and halved, it gives 127/72. This is the mean amount: multiplied by the period, it yields the total 889/162; or 5 *bharas*, 9 hundred [*palas*] and 60/80 [of a hundred].

Example: Tell the price of the seventh conch; the first being worth six *panas*, and the rest increasing by a *pana*?

Statement: Init. 6; Diff. 1; Per. 7. Answer: 12.

Example: A man gave his son-in-law sixteen *panas* the first day; and diminished the present by two a day. If thou be conversant with progression, say how many had he bestowed when the ninth day was past?

Statement: Init. 16; Diff. 2; Per. 9. Answer: 72; received by the son-in-law: or 72 the father-in-law's; being his disbursement.

1. The first term, common increase, and total amount, being known, to find the period. – Ch.

Example: Say how many are the layers in a stack containing a hundred bricks, and having at the summit ten, and increasing by five.

Statement: Init. 10; Com. diff. 5; Per.? Sum 100.

Operation: Twice the initial, 20, less the increase 5, is 15; the square of which is 225. The sum 100, eight (8) and increase 5, multiplied together, make 4,000. Add to this the square of the remainder, 225, the total is 4,225. Its square-root 65, less the foregoing remainder 15, gives 50; which divided by twice the common increase, 10, yields the period 5.

So in other cases likewise.

2. Triangles are three; tetragons five, and the circle is the ninth plane figure. Thus triangles are (*sama-tribhuja*) equilateral, (*dwi-sama-tribhuja*) isosceles, and (*vishama-tribhuja*) scalene. Tetragons are (*sama-chaturasra*) equilateral; (*ayata-sama-chaturasra*) oblong with equal sides [two and two]; (*dwi-sama-chaturasra*) having two equal sides; (*tri-sama-chaturasra*) having three sides equal; (*vishama-chaturasra*) having all unequal. – Ch.

3. *Bahu-pratibahu*, or *bhuja-pratibhupa* (§ 23): opposite sides.

4. Example: What is the area of an equilateral triangle, the side of which is twelve?

Statement:

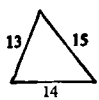
The sum of sides and of countersides, 12 and 24; their moieties 6 and 12; the product of which is 72, the gross area.

Example: What is the area of an isosceles triangle the base of which is ten and the sides thirteen?

Statement:

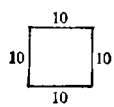
The moieties of the sums of opposite sides, 5 and 13; their product 65, the gross area.

Example: What is the area of a scalene triangle, the base of which is fourteen and the sides thirteen and fifteen?

Statement: 

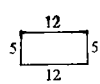
Answer: 98 the gross area.

Example: What is the area of an equilateral tetragon, the side of which is ten?

Statement: 

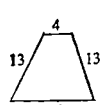
Answer: 100, the gross as well as exact area.

Example: What is the area of an oblong, two sides of which are twelve; and two, five?

Statement: 

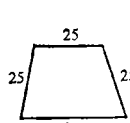
Answer: 60, the gross and exact area.

Example: What is the area of a quadrilateral having two equal sides thirteen, the base fourteen, and the summit four?

Statement: 

Answer: 117 the gross area.

Example: Tell the area of a quadrilateral having three equal sides twenty-five, and base thirty-nine?

Statement: 

Answer: 800 the gross area.

down four times, and severally lessened by the sides,[1] being multiplied together, the square-root of the product is the exact area.[2]

22. The difference of the squares of the sides being divided by the base, the quotient is added to and subtracted from the base: the sum and the remainder, divided by two, are the segments. The square-root, extracted from the difference of the square of the side and square of its corresponding segment of the base, is the perpendicular.

30–31. Assuming two triangles within the trapezium, let the diagonals be the bases of them.[3] Then the segments, separately

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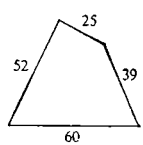
Example: Tell the gross area of a trapezium, of which the base is sixty, the summit twenty-five, and the sides fifty-two and thirty-nine?

Statement:



Answer: 1933¾ the gross area. – Ch.

1. The sides of the quadrilateral are severally subtracted from the half of the sum in all four places; but the sides of the triangle are subtracted in three, and the fourth remains as it stood. – Ch.

2. Examples as above. Sides of the equilateral triangle 12; the sum 36; its half set down four times 18, 18, 18, 18; which severally lessened by the sides gives 6, 6, 6, 18. The product of those numbers is 3,888, the surd root of which is the exact area.

Sides of the isosceles triangle 10, 13, 13; the sum 36. Its half 18, lessened severally by the sides, gives 5, 5, 8, 18. The product whereof is 3,600. The square-root of this is the exact area, 60.

Sides of the scalene triangle 14, 13, 15. Half the sum 21, less the sides, gives 7, 8, 6, 21. Product 7,056; the root of which is the exact area 84.

The gross area of the equilateral tetragon, as of the oblong, is the same with the exact area.

Sides of the tetragon with two equal sides, 14, 13, 13, 4. The exact area, as found by the rule, is 108.

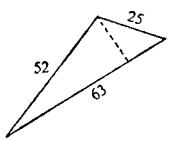
Sides of the tetragon having three equal sides, 39, 25, 25, 25. Exact area 768.

Sides of the trapezium 60, 52, 39, 25. Exact area 1,764. – Ch.

3. The greater diagonal is the base of one; and the summit and greater flank are its sides. The least diagonal is the base of the other; and the summit and least flank are the sides. – Ch.

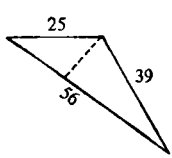
found, are the upper and lower portions formed by the intersection of the diagonals.[1] The lower portions of the two diagonals are taken for the sides of a triangle; and the base [of the tetragon] for its base. Its perpendicular is the lower portion of the [middle] perpendicular of the tetragon: the upper portion of it is the moiety of the sum of the [extreme] perpendiculars less the lower portion.[2]

1. In the tetragon just now instanced, the scalene triangle with the greater diagonal for base is this



The segments of its base as found by the rule (§22) are 48 and 15. These are respectively the lower and upper portions of the greater diagonal.

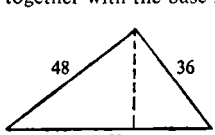
The scalene triangle with the less diagonal for base is



Here the segments, by the same rule (§ 22), are 36 and 20. They are the lower and upper portions of the least diagonal.

Or find the segments of one only: the perpendicular, found by the rule (§22) is the upper portion of the second diagonal: and subtracting that from the entire length, the remainder is the lower portion of it. Thus, in the foregoing example, the least segment in the first triangle is 15. Its square 225, subtracted from the square of the least side 625, leaves 400, the root of which is 20. It is the upper portion of the smaller diagonal, and subtracted from the whole length 56, leaves the lower portion 36. – Ch.

2. In the same figure, the scalene triangle composed of the two lower segments of the diagonals together with the base is this



Here the perpendicular found by the rule (§22) is $28\frac{4}{5}$. It is the lower portion

32.[1] At the intersection of the diagonals and perpendiculars, the lower segments of the diagonal and of the perpendicular are found by proportion: those lines less these segments are the upper segments of the same. So in the needle[2] as well as in the (*pata*) intersection [of prolonged sides and perpendiculars].[3]

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of the mean perpendicular. The greatest and least perpendiculars being $44\frac{4}{5}$ and $37\frac{4}{5}$, the moiety of their sum is $41\frac{3}{10}$. This is the length of the entire mean perpendicular. Subtracting from it its lower segment the residue is its upper segment $12\frac{1}{2}$. – Ch.

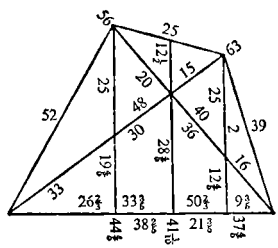
1. A rule to find the upper and lower portions of the diagonals and perpendiculars cut by the intersection of diagonals and perpendiculars, within a trapezium; also the lines of the needle and a figure of intersection.

2. *Suchi*, the needle; the triangle formed by the produced flanks of the tetragon. The section of a cone or pyramid.

Pata, sampata, tripata, intersection; of a prolonged side and perpendicular. The figure formed by such intersection.

3. Example: in a trapezium the base of which is sixty; one side fifty-two; the other thirty-nine; and the summit twenty-five: the greater diagonal sixty-three; the less, fifty-six: the greater perpendicular forty-five less one-fifth; its segments of the base, the greatest thirty-three and three-fifths, the least twenty-six and two-fifths: the least perpendicular thirty-seven and four-fifths; its segments of the base, greatest fifty and two-fifths, least nine and three-fifths: the perpendicular passing through the intersection of the diagonals, forty-one and three-tenths; its segments of the base, greatest thirty-eight and two-fifths, least twenty-one and three-fifths; tell the upper and lower portions of the perpendiculars, the intersections [of prolonged sides and perpendiculars] and the needle.

Here at the intersection of the diagonals, the segments of the greater diagonal, found as before (§ 30), are 48 and 15; those of the less are 36 and 20.



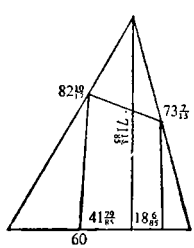
At the junction of the greater diagonal and greater perpendicular; the proportion is as diagonal sixty-three to the complement fifty and two-fifths,

so, to the segment twenty-six and two-fifths, what? or rendered homogeneous, $\frac{252}{5}$ | 63 | $\frac{132}{5}$. Answer: 33. It is the lower portion of the diagonal. Again, as the same complement is to the least perpendicular, so is the above mentioned segment to what? Statement: $\frac{252}{5}$ | $\frac{189}{5}$ | $\frac{132}{5}$. Answer: $19\frac{4}{5}$. It is the lower portion of the perpendicular. Substracting these from the whole diagonal 63 and entire perpendicular $44\frac{4}{5}$, the remainders are the upper segments of the diagonal and perpendicular; 30 and 25.

Next, at the junction of the less diagonal and less perpendicular: as the complement thirty-three and three-fifths is to the diagonal fifty-six, so is the segment nine and three-fifths to what? Statement: $33\frac{3}{5}$ | 56 | $9\frac{3}{5}$. Answer: 16, the lower portion of the diagonal. So, putting the perpendicular for the middle term, the lower portion of the less perpendicular comes out $12\frac{4}{5}$. By subtraction from the entire diagonal and perpendicular, their upper segments are obtained 40 and 25.

In like manner, for any given question, the solution may be variously devised with the segment of the base for side, the segment of the perpendicular for upright, and the segment of the diagonal for hypotenuse.

The operation on the needle is next exhibited



The segments of the base on either side of the perpendicular let fall from the top of the needle come out $41\frac{79}{85}$ and $18\frac{6}{85}$ (The text relative to the method of finding these segments is irretrievably corrupt; and has been therefore omitted in the version.) With either of these segments the mean perpendicular is found by proportion: if the least segment $9\frac{3}{5}$ give the least perpendicular $37\frac{4}{5}$, what does the segment $18\frac{6}{85}$ give? Answer $71\frac{13}{85}$. It is the perpendicular let fall from the summit of the needle. In the same manner, with the greater segment, the same length of the perpendicular is deduced.

Next, to find the sides of the needle: As the least perpendicular is to the side thirty-nine, so is the middle perpendicular to what? Statement $37\frac{4}{5}$ | 39 | $71\frac{13}{85}$. Answer: $73\frac{7}{17}$. Or the side may be found from the segments: thus $9\frac{3}{5}$ | 39 | $18\frac{6}{85}$. Answer $73\frac{7}{17}$ as before. To find the greater side: As the greater perpendicular is to the side fifty-two, so is the perpendicular of the needle to what? $44\frac{4}{5}$ | 52 | $71\frac{13}{85}$. Answer: $82\frac{10}{17}$. Or proportion may be taken with the segments of the base: $26\frac{2}{5}$ | 52 | $41\frac{79}{85}$. Answer $82\frac{10}{17}$, as before. See figure [as above].

39.[1] The height of the mountain, taken into a multiplier arbitrarily put, is the distance of the town. That result being reserved, and divided by the multiplier added to two, is the height of the leap. The journey is equal.[2]

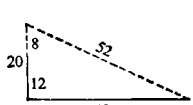
41. In a circle the chord is the square-root of the diameter less the arrow taken into the arrow and multiplied by four.[3] The

1. Within an oblong tetragon, to describe a figure such that the sum of the side and one portion of the upright may be equal to the diagonal and remaining portion of the upright: so as the journeys may be equal. – Ch. See *Lilavati*, § 154, and *Vija-ganita*, § 126; where the same problem is introduced: substituting, however, in the example, a tree, an ape and a pond, for a hill, a wizard and a town.

2. Example: on the top of a certain hill live two ascetics. One of them, being a wizard, travels through the air. Springing from the summit of the mountain, he ascends to a certain elevation, and proceeds by an oblique descent, diagonally, to a neighbouring town. The other, walking down the hill, goes by land to the same town. Their journeys are equal. I desire to know the distance of the town from the hill, and how high the wizard rose.

This being proposed, the rule applies; and its interpretation is this; any elevation of the mountain is put; and is multiplied by an arbitrarily assumed multiplier: the product is the distance of the town from the mountain. Then divide this reserved quantity by the multiplier added to two, the quotient is the number of *yojanas* of the wizard's ascent. The sum of the hill's elevation and wizard's ascent is the upright; the distance of the town from the mountain is the side: the square-root of the sum of their squares is the diagonal (hypotenuse): it is the oblique interval between the town and the summit of the rise.

Thus, let the height of the mountain be twelve. This, multiplied by an arbitrarily assumed multiplier four, 12 by 4, makes 48. It is the distance of the town from the hill. This divided by the multiplier added to two, 48 by 6, gives 8. It is the ascent. Here the upright is 20: its square is 400. The side is 48; the square of which is 2304. The sum of these squares is 2,704; and its square-root 52. The semirectangle is thus found.



Here also the sum of the side and lower portion of the upright is 60, the journey of one of the ascetics: and the upper portion added to the hypotenuse is that of the other, likewise 60.

3. Example: Within a circle, the diameter of which is ten, in the place where the arrow is two, what is the chord?

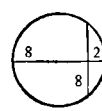
square of the chord divided by four times the arrow, and added to the arrow, is the diameter.[1]

CHAPTER XVIII

Section II: Algorithm[2]

31. Rule for addition of affirmative and negative quantities and cipher: § 19. The sum of two affirmative quantities is affirmative;

Diameter 10: less the arrow 2; remains 8. This multiplied by the arrow makes 16; which multiplied by 4, gives 64: the square-root of which is 8. See figure.

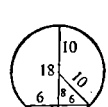


The principle of the rule for finding the square of the chord (in the construction of tabular sines) is here to be applied. But the square is in this place multiplied by four, because the entire chord is renbired. – Ch.

1. Example: Chord 8. Its square 64, divided by four times the arrow 2, viz. 8; gives the quotient 8: to which adding the arrow, the sum is 10.

Example 2d: A bambu, eighteen cubits high, was broken by the wind. Its tip touched the ground at six cubits from the root. Tell the length of the segments of the bambu.

Statement: Length of the bambu 18. It is the diameter less the least arrow. (What is termed by us 'diameter less the arrow', is by Āryabhata denominated the greater arrow. For he says, 'In a circle the product of the arrows is equal to the square of the semichord of both arcs.') The ground from the root, to the point where the tip fell, is 6: it is the semichord. Its square is 36. This is equal to the diameter less the arrow multiplied by the arrow. Dividing it by the diameter less the arrow, viz. 18, the quotient is 2. It is the arrow. Adding this to the diameter less the arrow, the sum is the diameter, 20. Half of this, 10, is the semidiameter. It is the upper portion of the bambu and is the hypotenuse. Subtracted from eighteen, it leaves the upright, or lower portion of the bambu, 8. The side is the interval between the root and tip, 6. The point of fracture of the bambu is the centre of the circle. See figure



2. *Shat-trinsat-paricarman.* Thirty-six operations or modes of process. See Arithm. § 1. *Vijgan.* § 3.

of two negative is negative; of an affirmative and a negative is their difference; or, if they be equal, nought. The sum of cipher and negative is negative; of affirmative and nought is positive; of two ciphers is cipher.

32–3. Rule for subtraction: § 20–1. The less is to be taken from the greater, positive from positive; negative from negative. When the greater, however, is subtracted from the less, the difference is reversed. Negative, taken from cipher, becomes positive; and affirmative becomes negative. Negative, less cipher, is negative; positive, is positive; cipher, nought. When affirmative is to be subtracted from negative, and negative from affirmative, they must be thrown together.

34. Rule for multiplication: § 22. The product of a negative quantity and an affirmative is negative; of two negative, is positive; of two affirmative, is affirmative. The product of cipher and negative, or of cipher and affirmative, is nought; of two ciphers, is cipher.

35–6. Rule for division: § 23–4. Positive, divided by positive or negative by negative, is affirmative. Cipher, divided by cipher, is nought. Positive, divided by negative, is negative. Negative, divided by affirmative, is negative. Positive, or negative, divided by cipher, is a fraction with that for denominator:[1] or cipher divided by negative or affirmative.[2]

[36 Concluded.] Rule for involution and evolution: § 24. The square of negative or affirmative is positive; of cipher, is cipher. The root of a square is such as was that from which it was [raised].[3]

78. Rule: To find a quantity such, that, being severally multiplied by two multipliers, and having unity added in each instance, both sums may afford square roots: § 46. The sum of the multiplier, being multiplied by eight, and divided by the square of

1. *Tach-ch heda*, having that for denominator: having, in this instance, cipher for denominator, to a finite quantity for numerator. See *Vij.-gan.* §16.

2. Is in like manner expressed by a fraction having a finite denominator to a cipher for numerator.

3. The root is to be taken either negative or affirmative, as best answers for the further operations. – Com-

their difference, is the quantity [sought]. The two multipliers, tripled and added to the opposite, and divided by the difference, are the roots.[1]

79. Question 33. The residue of seconds of the moon, severally multiplied by seventeen, and by thirteen, and having one added, [becomes in both instances] a square. Solving [this problem] within a year [the proficient is] a mathematician.

Here the multipliers are 17 and 13. Their sum 30: multiplied by eight, 240. Difference of the multipliers 4: its square 16. Quotient of the division, 15: it is the number [sought]; and it is the residue of the seconds of the moon.

To find the roots: multipliers 17, 13; multiplied by three, 51, 39: added to the reciprocals, 56, 64. Divided by the difference of the multipliers 4, the roots come out 14, 16.

80. Rule: § 47. A square, with another square added and subtracted, being multiplied by the quotient of the sum of that sum and difference divided by the square of half their difference, produces numbers, of which both the sum and difference are squares; as also the product with one added to it.[2]

81. Question 34. The residue of minutes of the sun on Wednesday, having the residue of seconds on Thursday added and subtracted, yields in both instances an exact square; and so does the product with one added. A person solving [this problem] within a year is a mathematician.

Here let an assumed square be 16; with another square, as 4, added and subtracted, 20 and 12: sum of these 32. Divided by the square of half the difference of these quantities, namely 16, the

1. The proposed multipliers are to be added together: and the sum, being multiplied by eight, and divided by the square of the difference of these multipliers, is the quantity [sought]. How are the roots found? The author proceeds to reply: multiplying the multipliers severally by three, add to the two products the opposite multiplier respectively. Then dividing by the difference of the multipliers, the quotients are the roots. – Com.

2. Some square of an arbitrary number is to be set down, and the square of another arbitrary number is to be added in one place and subtracted in another. The sum of these two quantities is divided by the square of half their difference: the quotient is their multiplier. Multiplied by it, they are the numbers sought: of which if the sum be taken, it is a square; if the difference, it also is square; if the product with unity added, this again is square. – Com.

quotient is their multiplier, 2. Multiplied by it, the two quantities come out 40, 24. The first is residue of minutes of the sun, 40. Hence, as before, the number of [elapsed] days is deduced, 3,385. The second is residue of seconds of the sun, 24: whence the numbers of days, 27. Adding five times the divisor, 5,480, the number of [elapsed] days on Thursday comes out 5,509. So, by virtue of suppositions, manifold answers may be obtained.

82. Rule: To find a quantity, such that having two given numbers added, or else subtracted, the results may be exact squares: § 48. The difference of the numbers, by addition or subtraction of which the quantity becomes a square, is divided by an arbitrary number and has it added or subtracted: the square of half the result, having the greater number added or subtracted, is the quantity which answers in the case of addition or subtraction.[1]

83. Question 35. Making the residue of minutes of the sun on a Wednesday, with the addition of twelve and of sixty-three, and with the subtraction of sixty and of eight, an exact square; [the proficient solving the problem] within a year is a mathematician.

Two questions are here proposed. The numbers, which are to be added to the quantity, are separated, 12, 63. Their difference, 51; divided by an arbitrary number, as 3, gives 17; with the same added, since addition is in question, the sum is 20; its moiety, 10; the square of which is 100. The greater of the two additive quantities is 63. Subtracting this, the result is the quantity sought, 37. With either twelve or sixty-three added, it is an exact square.

1. Of the two quantities, the addition of which makes the quantity in question an exact square, or the subtraction of which does so, the difference is in every case to be taken. This step is common to both methods. Dividing the difference then by an arbitrary number; the quotient must have added to it the same arbitrary number; if addition were given by the question: but, if subtraction were so, the same quotient must have the arbitrary number subtracted. Then the quantity resulting in either case is to be halved; and the half, to be squared. [From which subtracting the greater number, the remainder is the quantity which answers. (The original is deficient: but may be thus supplied from comparison of the text, and of the example as wrought.)] If the condition were addition: but, if it were subtraction, the square of the moiety appertaining to the case being added to the greater number, the sum is the quantity sought. – Com.

In the example of subtraction; the two numbers which are to be subtracted to make a square, are 60, 8. Their difference, divided by an arbitrary number, namely two, yields 26: less the arbitrary number, leaves 24: its moiety 12; the square of which is 144. Here the greater of the two subtractive quantities is 60. This added to the square is 204. It is the quantity, which lessened by sixty, affords a square root; or by eight. It is the residue of minutes of the sun. Hence, as before, the number of [elapsed] days on Wednesday, is to be deduced.

In like manner, by virtue of suppositions, manifold answers may be obtained.

84. Rule: § 49. The sum of the numbers, the addition and subtraction of which makes the quantity a square, being divided by an arbitrarily assumed number, has that assumed number taken from the quotient: the square of half the remainder, with the subtractive number added to it, is the quantity [sought].[1]

85. Question 36. Making the residue of seconds of the sun on Wednesday, with ninety-three added, or with sixty-seven subtracted, an exact square, [a proficient solving this problem] within a year [is] a mathematician.

Here the subtractive number is 67; the additive number, 93; their sum, 160: divided by an assumed number 4, makes 40: less the assumed number, leaves 36; the half of which is 18: its square, 324: added to the subtractive quantity 67, the quantity is found 391. It is the residue of seconds of the sun.

1. If a pair of quantities equal or unequal be given such, that a quantity, which lessened by the first, is an exact square, is also a square when increased by the second; then the two proposed quantities are to be added together; and their sum is to be divided by some arbitrary number. From the quotient subtracting the same arbitrary number, the half of the remainder is taken: and the square of that moiety, added to the number the subtraction of which renders the quantity in question a square, is in every case the quantity sought. – Com.

15

HSIA-HOU YANG

(Sixth century)

HSIA-HOU YANG, one of the three greatest Chinese mathematicians of the sixth century, was the author of one of the most practical of the older Chinese mathematical works, the *Hsia-hou Yang Suan-ching* ('Arithmetical Classic of Hsia-hou Yang'). This treatise became one of the standard texts on the required list for the examination of civil servants during the T'ang Dynasty (618–906). Internal evidences in his writings indicate that Hsia-hou Yang wrote and taught his arithmetic in the middle or late sixth century. The *Hsia-hou Yang Suan-ching* was based on previous Chinese mathematical classics, primarily upon Chang Tsang's *K'iu-ch'ang Suan-shu* ('Arithmetical Rules in Nine Sections') of the second century B.C., and upon the *Sun-Tsu Suan-ching* ('Arithmetic of Sun-Tsu') of the third century A.D. Divided into numerous small items, it was embodied in the great fifteenth-century Chinese encyclopaedia, *Yung-lo Ta-tien.* In the eighteenth century, the learned Tai Chen of Peking collated the various entries in the *Yung-lo Ta-tien* which had been taken from the *Hsia-hou Yang Suan-ching.* Making use of the index of the encyclopaedia from which it was possible to reconstruct the arrangement of the original material, he painstakingly reconstituted and edited Hsia-hou Yang's famous work. Tai Chen's edition, the source of our modern version, was published in 1776 with the encouragement and financial support of the Emperor Khien-Lung who honoured the publication by writing a preface to it.

The original work is said to have been concisely put in a total of twelve thousand words on thirty-six double pages. Hsia-hou Yang laid great stress on usefulness, and accordingly he included only the material needed in the solution of commercial problems or in the business of government. No rules were given. Occasionally some explanation was briefly stated, but in the main, each problem was accompanied merely by the answer. The problems were varied. Some were on mensuration, others were concerned with taxes, with the computation of commissions, and even with questions regarding the division of loot and food by

army officers among the soldiers. The problems were such as required no more than elementary calculations involving fractions, square and cube roots, and percentage.

ARITHMETICAL CLASSIC OF HSIA-HOU YANG

by Pere Louis Vanhee, S.J.

In the first section the five operations of addition, subtraction, multiplication, division and square and cube roots are given. The work on division is subdivided into (1) 'ordinary division'; (2) 'division by ten, hundred, and so on', especially intended for work in mensuration; (3) 'division by simplification' (*yo ch'u*). The last problem in the section is as follows:

'There are 1843 *k'o*, 8 *t'ow*, 3 *ho* of coarse rice. A contract requires that this be exchanged for refined rice at the rate of 1 *k'o*, 4 *t'ow* for 3 *k'o*. How much refined rice must be given?' The answer is 860 *k'o*, 534 *ho*. The solution is given as follows: 'Multiply the given number by 1 *k'o*, 4 *t'ow* and divide by 3 *k'o* and you will obtain the result'.[1]

Fractions are also mentioned, special names being given to the four most common ones, as follows:

1/2 is called *chung p'an* (even part);
1/3 is called *shaw p'an* (small part);
2/3 is called *thai p'an* (large part);
1/4 is called *joh p'an* (weak part).

In the second section there are twenty-eight applied problems relating to taxes, commissions, and such questions as concern the division by army officers of loot and food (silk, rice, wine, soy sauce, vinegar and the like) among their soldiers.

The third section contains forty-two problems each beginning with the word 'now', which is here taken as substantially equivalent to the word 'if'. Five of these problems, translated as literally as possible, are as follows:

Ex. 1. Now for 1 pound of gold one gets 1,200 pieces of silk. How many can you get for 1 ounce? Answer: For 1 ounce you

1. 1 *k'o* = 10 *t'ow* = 100 *ho*. The *k'o* may be roughly translated as a bushel.

get exactly 75 pieces. Solution: Take the given number of pieces, have it divided by 16 ounces, and you will obtain the answer.[1]

Ex. 2. Now you have 192 ounces of silk. How many *choo* have you? Answer: Four thousand six hundred and eight.[2]

Ex. 15. Now 2,000 packages of cash must be carried to the town at the rate of 10 cash per bundle. How much will be given to the mandarin and how much to the carrier? Answer: 1,980 packages and 198-2/101 cash to the mandarin; 19 packages and 801-98/101 to the carrier. Solution: Take the total number as the dividend and 1 package plus 10 cash as the divisor.[3]

Ex. 24. Out of 3,485 ounces of silk how many pieces of satin can be made, 5 ounces being required for each piece? Answer: 697. Solution: Multiply the number of ounces by 2 and go back by one row. Dividing by 5 will also give the answer.[4]

Ex. 42. Now they build a wall, high 3 rods, broad 5 feet at the upper part and 15 feet at the lower part; the length 100 rods. For a 2-foot square a man works 1 day. How many days are required? Answer: 75,000. Solution: Take half the sum of the upper and lower breadths, have it multiplied by the height and length; the product will be the dividend. As the divisor you will use the square of the given 2 feet.[5]

1. The Chinese pound was, from early times, divided into 16 ounces or *taels*.

2. The result (4,608) is written in words instead of in numeral characters. The zero was not in use in the sixth century; but this did not, of course, prevent the use of the ordinary Chinese numerals of that period. The answer shows that the ounce was at that time divided into 24 *choo*.

At the beginning of most of the early Chinese books on arithmetic there is the following statement from Sun-tzi's *Suan-king*, a work referred to in Smith's *History of Mathematics*, vol. I, p. 141: 'Weight begins with one grain of millet; ten grains make one *ts'en*; ten *ts'en* make one *choo*; 24 *choo* make one ounce (*tael*); 16 ounces make one pound; 30 pounds make one *kiun*; 4 *kiun* make 1 stone.'

3. A package or string of cash contains 1,000 farthings. The two results reduce to periodic fractions. There is no reason given for the division of 2,000,000 by 1,010.

4. This is an early use of our rule for dividing by 5. The expression about going back one row seems to refer to the use of counters. So far as known, the *suan-pan*, in its present form, was not yet invented.

5. 1 rod = 10 feet = 100 inches. The amount made by one man in a day is evidently intended to be the volume of a rectangular solid of base 2 feet square and of height 1 rod; that is, 40 cubic feet. The divisor, therefore, is 40.

Hsia-Hou Yang also uses percentage. He shows considerable ability in finding various areas and volumes. His work is evidently a good type of practical textbook of the time, a fact that is shown by its popularity and the high esteem in which it has always been held. Yuan Yuan,[1] for example, in his well-known biographical work, speaks of it as 'an easy text, intended for daily use'.

1. See Smith, loc. cit., vol. I, p. 535. Yuan Yuan was born in 1764 and died in 1849.

16

WANG HS'IAO-T'UNG

(Seventh century)

WANG HS'IAO-T'UNG, Chinese astronomer and mathematician, author of the *Ch'i-ku Suan-ching* ('Arithmetical Classic of Ancient Rules'), which despite its name presented a significant advance over the ancient methods, belonged to the early period of the T'ang Dynasty Few facts of his life are known beyond Wang's service in the T'ang government as an expert on the Astronomical Board. It was his duty in 623, 626 and so on, as a Doctor of Arithmetic, to subject to critical examination the contents of the calendar composed by Fu Jen-Chun in 618. We know something of his personality from his writing which reveals a vast self-assurance and supreme confidence in his work.

Wang Hs'iao-t'ung's mathematical treatise, *Ch'i-ku Suan-ching*, has come down to us in very nearly complete form. Only a few lines have been lost where part of the last page is torn away. The treatise consists of a set of twenty problems in solid mensuration and Wang's solutions and commentaries on them. The problems were originally intended to illustrate the fifth section of the old 'Nine Sections'. Many of the solutions are such as involve cubic equations. Attached to the work and forming a kind of preface to it, is a letter of presentation written by Wang to his Emperor stating that the methods were of his own devising. In a challenge to his readers, Wang offered a thousand taels of silver to anyone who could find a single error in his work. The *Ch'i-ku Suan-ching* by Wang Hs'iao-t'ung antedated by about six hundred years the next mention of cubic equations in Chinese mathematical writings.

CH'I-KU SUAN-CHING

From The Development of Mathematics in China and Japan
by Yoshio Mikami

'There is a right triangle, the product of whose two sides is $706\frac{1}{50}$, and whose hypotenuse is greater than the first side by $30\frac{9}{60}$. It is required to know the lengths of the three sides.'

Wang gives the answer of this problem to be $14\frac{7}{20}$, $49\frac{1}{5}$ and $51\frac{1}{4}$. He then proceeds to give the rule:

'The product (P) being squared and being divided by twice the surplus (S), make the result the *shih* or the constant class. Halve the surplus, and make it the *lien-fa* or the second degree class. And carry out the operation of evolution according to the extraction of cube root. The result gives the first side. Adding the surplus to it, one gets the hypotenuse. Divide the product with the first side and the quotient is the second side.'

The rule is the same as to construct the cubic equation

$$x^3+\frac{S}{2}x^2-\frac{P^2}{2S}=0,$$

or the arrangement as annexed, and to solve it in a process similar to the extraction of a cube root. It is very evident that the way of solving an equation of the third degree in this way applies only to numerical equations.

$-\frac{P^2}{2S}$	0^0
0	1^0
$\frac{S}{2}$	2^0
1	3^0

'In a right triangle, the product of the first side and the hypotenuse is $397\frac{3}{4}$, and the difference of the hypotenuse and the second side is $\frac{1}{10}$; what is the length of this second side?

'Answer: $92\frac{2}{5}$.

'Rule. The (given) product (P) being squared and divided by twice the (given) difference (D), we call the result the "solid volume". Again cube the (given) difference, and halve it, when the difference of this quantity and the "solid volume" should be taken for the *shih* or the constant class. Again square the (given) difference and take its twice as the *fang-fa* or the 1st degree class. Again take $\frac{5}{2}$ of the (given) difference as the *lien-fa* or the 2nd degree class. Then (making 1 the 3rd degree class) divide according to the rule of the cubic root extraction. The root gives the length required.'

This rule is equivalent to the construction of the cubic equation,

$$x^3+\frac{5}{2}Dx^2+2D^2x=\frac{P^2}{2D}-\frac{D^3}{2}$$

'In a store-house shaped like a truncated square pyramid, the difference of whose upper and lower sides is 6 feet (D) and whose height is 9 feet (G) greater than the upper side, there is contained 187·2 measures of corn, of which 50·4 measures are carried out. What will be the lengths of the upper and lower sides of the store-house, its height, and the depth and a side of the upper end of the remaining corn?

'Answer: Upper side is 3 feet; lower side is 9 feet; height 12 feet; depth and upper side of the remainder are each 6 feet.'

To find the upper side of the store-house in this problem, Wang Hs'iao-t'ung applies the equation,

$$x^3+(D+G)x^2+\left(DG+\frac{D^2}{3}\right)x=P-\frac{D^2G}{3},$$

where P represents the volume of the store-house expressed in cubic feet. This is the very equation to which we are led by eliminating y and z from

$$y-x=D,\quad z-x=G,\quad \tfrac{1}{3}z(xy+x^2+y^2)=P,$$

where y and z represent the lower side and the height respectively and the last equation is in accordance with the formula that appears in the 'Arithmetic in Nine Sections'.

Wang Hs'iao-t'ung gives in the *Ch'i-ku Suan-ching* only his rules for the arrangement of the equations; he does not give the particulars of the procedure for arriving at such equations.

17

MOHAMMED BEN MUSA AL-KHOWARIZMI

(*c.* 825)

MOHAMMED BEN MUSA AL-KHOWARIZMI (also, Abu Abdallah Muhammed ibn Musa al-Khwarizmi), son of Moses, the Khowarezmite, was born in the city of Khowarezm (modern Khiva). He was one of the greatest of the scientists who were encouraged by the Caliph Al-Mamun to gather at his court at Baghdad in the early ninth century and to participate in the work of his newly organized *Bayt al-hikma* (House of Wisdom). Al-Mamun constructed two astronomical observatories for his new academy and he made strenuous efforts to obtain extant scientific works for it. In his eagerness to collect scientific manuscripts, he went so far as to send a mission to the Byzantine Emperor Leon for the purpose of securing Greek works. The writings collected by Al-Mamun were immediately translated into Arabic by the scholars of his academy. It was in this scholarly atmosphere of the *Bayt al-hikma* that al-Khowarizmi worked and reached the peak of his activities as an astronomer, mathematician and geographer. He gained fame early through his abridgement (known as the less *Sind-hind*) of al-Fazan's Arabic translation (the greater *Sind-hind*) of the first Hindu astronomical tables to be brought to the Arabian court. Al-Khowarizmi's less *Sind-hind*, a popular work from the first, was followed by an enlarged version containing observations made by al-Khowarizmi himself, and including some material possibly based on Ptolemy. The fact that the enlarged work remained in use for centuries after his death is a small part of the evidence of al-Khowarizmi's influence on the scholars who followed him. Arabian and European writers continued to explain al-Khowarizmi's works for hundreds of years. His trigonometric tables, revised and enlarged by Meslama al-Majriti (about A.D. 1000) were translated into Latin by Adelhard of Bath (about 1126).

Exceptional historical interest attaches to two of his mathematical works, one on arithmetic, the other on algebra. The treatise on arithmetic which had taught the methods of Hindu computation to the

Arabs served a like purpose for the European scholars. A Latin translation of this work entitled *Algoritmi de numero Indorum* appeared about 1120. The Latinized version of his name in the opening words *Dixit Algoritmi* (Algorithm says) made his name synonymous with the art of computation, giving rise to the term *augrim* in Chaucer's *A Treatise on the Astrolabe*, and to the modern word algorism (or algorithm). The works of Alexander de Villa Dei (*c.* 1220) and of John Halifax (Sacrobosco), drawing upon al-Khowarizmi's arithmetic, spread the knowledge of the Hindu system in Europe. Sacrobosco's *Algorismus vulgaris*, written about 1250, was widely used as late as the sixteenth century.

Al-Khowarizmi's algebra *Al-jabr w'al-mugabala* contained the first systematic treatment of the subject. The very name of the branch of mathematics to which the treatise is devoted 'algebra' is derived from the title of this work. In the complete absence of axiomatic foundations and of proofs depending upon them, and in the analytical presentation of its solutions, al-Khowarizmi's algebra evidences the author's marked preference for Hindu over Greek methods. Nevertheless, the content shows a blending of Eastern and Greek achievements. Through the persistent reappearance in subsequent algebra texts of certain features of his work, such as the particular equations used, or the numerical problems cited, or the order in which the equations or problems are discussed, al-Khowarizmi's influence upon writers in the field of algebra can be traced directly or indirectly for more than seven hundred years after his death.

THE ALGEBRA OF MOHAMMED BEN MUSA

translated from the Arabic, with explanatory notes by Frederic Rosen

THE AUTHOR'S PREFACE

IN THE NAME OF GOD, GRACIOUS AND MERCIFUL!

This work was written by Mohammed Ben Musa, of Khowarezm. He commences it thus:

Praised be God for his bounty towards those who deserve it by their virtuous acts: in performing which, as by him prescribed to his adoring creatures, we express our thanks, and render ourselves worthy of the continuance (of his mercy), and preserve our-

selves from change: acknowledging his might, bending before his power, and revering his greatness!

MOHAMMED BEN MUSA'S COMPENDIUM ON CALCULATING BY COMPLETION AND REDUCTION

When I considered what people generally want in calculating, I found that it always is a number.

I also observed that every number is composed of units, and that any number may be divided into units.

I observed that the numbers which are required in calculating by Completion and Reduction are of three kinds, namely, roots, squares and simple numbers relative to neither root nor square.

A root is any quantity which is to be multiplied by itself, consisting of units, or numbers ascending, or fractions descending.[1]

A square is the whole amount of the root multiplied by itself.

A simple number is any number which may be pronounced without reference to root or square.

A number belonging to one of these three classes may be equal to a number of another class; you may say, for instance, 'squares are equal to roots', or 'squares are equal to numbers', or 'roots are equal to numbers'.[2]

Of the case in which *squares are equal to roots*, this is an example. 'A square is equal to five roots of the same';[3] the root of the square is five, and the square is twenty-five, which is equal to five times its root.

So you say, 'one-third of the square is equal to four roots';[4] then the whole square is equal to twelve roots; that is a hundred and forty-four; and its root is twelve.

Or you say, 'five squares are equal to ten roots';[5] then one square is equal to two roots; the root of the square is two, and its square is four.

In this manner, whether the squares be many or few (i.e.

1. By the word root, is meant the simple power of the unknown quantity.

2. $cx^2 = bx \quad cx^2 = a \quad bx = a.$

3. $x^2 = 5x \quad \therefore \; x = 5.$

4. $\frac{x^2}{3} = 4x \quad \therefore \; x^2 = 12x \quad \therefore \; x = 12.$

5. $5x^2 = 10x \quad \therefore \; x^2 = 2x \quad \therefore \; x = 2.$

multiplied or divided by any number), they are reduced to a single square; and the same is done with the roots, which are their equivalents; that is to say, they are reduced in the same proportion as the squares.

As to the case in which *squares are equal to numbers*; for instance, you say, 'a square is equal to nine';[1] then this is a square, and its root is three. Or 'five squares are equal to eighty';[2] then one square is equal to one-fifth of eighty, which is sixteen. Or 'the half of the square is equal to eighteen';[3] then the square is thirty-six, and its root is six.

Thus, all squares, multiples and sub-multiples of them, are reduced to a single square. If there be only part of a square, you add thereto, until there is a whole square; you do the same with the equivalent in numbers.

As to the case in which *roots are equal to numbers*; for instance, 'one root equals three in number';[4] then the root is three, and its square nine. Or 'four roots are equal to twenty';[5] then one root is equal to five, and the square to be formed of it is twenty-five. Or 'half the root is equal to ten';[6] then the whole root is equal to twenty, and the square which is formed of it is four hundred.

I found that these three kinds: namely, roots, squares and numbers, may be combined together, and thus three compound species arise;[7] that is, 'squares and roots equal to numbers'; 'squares and numbers equal to roots'; 'roots and numbers equal to squares'.

Roots and Squares are equal to Numbers;[8] for instance, 'one

1. $x^2 = 9 \quad x = 3.$
2. $5x^2 = 80 \quad \therefore \; x^2 = \frac{80}{5} = 16.$
3. $\frac{x^2}{2} = 18 \quad \therefore \; x^2 = 36 \quad \therefore \; x = 6.$
4. $x = 3.$
5. $4x = 20 \quad \therefore \; x = 5.$
6. $\frac{x}{2} = 10 \quad \therefore \; x = 20.$
7. The three cases considered are,
   1st $cx^2 + bx = a$ 2nd $cx^2 + a = bx$ 3rd $cx^2 = bx + a.$
8. 1st case: $cx^2 + bx = a.$
   Example $x^2 + 10x = 39$
   $x = \sqrt{[(\frac{10}{2})^2 + 39]} - \frac{10}{2}$
   $= \sqrt{64} - 5$
   $= 8 - 5 = 3.$

square, and ten roots of the same, amount to thirty-nine dirhems'; that is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine? The solution is this: you halve the number[1] of the roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought for; the square itself is nine.

The solution is the same when two squares or three, or more or less be specified;[2] you reduce them to one single square, and in the same proportion you reduce also the roots and simple numbers which are connected therewith.

For instance, 'two squares and ten roots are equal to forty-eight dirhems';[3] that is to say, what must be the amount of two squares which, when summed up and added to ten times the root of one of them, make up a sum of forty-eight dirhems? You must at first reduce the two squares to one; and you know that one square of the two is the moiety of both. Then reduce everything mentioned in the statement to its half, and it will be the same as if the question had been, a square and five roots of the same are equal to twenty-four dirhems; or, what must be the amount of a square which, when added to five times its root, is equal to twenty-four dirhems? Now halve the number of the roots; the moiety is two and a half. Multiply that by itself; the product is six and a quarter. Add this to twenty-four; the sum is thirty dirhems and a quarter. Take the root of this; it is five and a half. Subtract from this the moiety of the number of the roots, that is two and a half; the remainder is three. This is the root of the square, and the square itself is nine.

1. i.e. the coefficient.
2. $cx^2+bx=a$ is to be reduced to the form $x^2+\frac{b}{c}x=\frac{a}{c}$.
3. $2x^2+10x=48$
   $x^2+5x=24$
   $x=\sqrt{[(\frac{5}{2})^2+24]}-\frac{5}{2}$
   $=\sqrt{[6\frac{1}{4}+24]}-2\frac{1}{2}$
   $=5\frac{1}{2}-2\frac{1}{2}=3.$

The proceeding will be the same if the instance be, 'half of a square and five roots are equal to twenty-eight dirhems';[1] that is to say, what must be the amount of a square, the moiety of which, when added to the equivalent of five of its roots, is equal to twenty-eight dirhems? Your first business must be to complete your square, so that it amounts to one whole square. This you effect by doubling it. Therefore double it, and double also that which is added to it, as well as what is equal to it. Then you have a square and ten roots, equal to fifty-six dirhems. Now halve the roots; the moiety is five. Multiply this by itself; the product is twenty-five. Add this to fifty-six; the sum is eighty-one. Extract the root of this; it is nine. Subtract from this the moiety of the number of roots, which is five; the remainder is four. This is the root of the square which you sought for; the square is sixteen, and half the square eight.

Proceed in this manner, whenever you meet with squares and roots that are equal to simple numbers; for it will always answer.

Squares and Numbers are equal to Roots;[2] for instance, 'a square and twenty-one in numbers are equal to ten roots of the same square'. That is to say, what must be the amount of a square, which, when twenty-one dirhems are added to it, becomes equal to the equivalent of ten roots of that square? Solution: Halve the number of the roots; the moiety is five. Multiply this by itself; the product is twenty-five. Subtract from this the twenty-one which are connected with the square; the remainder is four. Extract its root; it is two. Subtract this from the moiety of the roots, which is five; the remainder is three. This is the root of the square which you required, and the square is nine. Or you may add the root to the moiety of the roots; the sum is seven; this is the root of the square which you sought for, and the square itself is forty-nine.

1. $\frac{x^2}{2}+5x=28$

$x^2+10x=56$

$x=\sqrt{[(\frac{10}{2})^2+56]}-\frac{10}{2}$

$=\sqrt{25+56}-5$

$=\sqrt{81}-5$

$=9-5=4.$

2. 2nd case. $cx^2+a=bx$

Example $x^2+21=10x$

$x=\frac{10}{2}\pm\sqrt{[(\frac{10}{2})^2-21]}$

$=5\pm\sqrt{25-21}$

$=5\pm\sqrt{4}$

$=5\pm2.$

When you meet with an instance which refers you to this case, try its solution by addition, and if that do not serve, then subtraction certainly will. For in this case both addition and subtraction may be employed, which will not answer in any other of the three cases in which the number of the roots must be halved. And know, that, when in a question belonging to this case you have halved the number of the roots and multiplied the moiety by itself, if the product be less than the number of dirhems connected with the square, then the instance is impossible;[1] but if the product be equal to the dirhems by themselves, then the root of the square is equal to the moiety of the roots alone, without either addition or subtraction.

In every instance where you have two squares, or more or less, reduce them to one entire square,[2] as I have explained under the first case.

Roots and Numbers are equal to Squares;[3] for instance, 'three roots and four of simple numbers are equal to a square'. Solution: Halve the roots; the moiety is one and a half. Multiply this by itself; the product is two and a quarter. Add this to the four; the sum is six and a quarter. Extract its root; it is two and a half. Add this to the moiety of the roots, which was one and a half; the sum is four. This is the root of the square, and the square is sixteen.

Whenever you meet with a multiple or sub-multiple of a square, reduce it to one entire square.

1. If in an equation, of the form $x^2+a=bx$, $\left(\frac{b}{2}\right)^2<a$, the case supposed in the equation cannot happen. If $\left(\frac{b}{2}\right)^2=a$, then $x=\frac{b}{2}$.

2. $cx^2+a=bx$ is to be reduced to $x^2+\frac{a}{c}=\frac{b}{c}x$.

3. 3rd case $cx^2=bx+a$

   Example $x^2=3x+4$

$$\begin{aligned} x^2 &= \sqrt{[(\tfrac{3}{2})^2+4]}+\tfrac{3}{2} \\ &= \sqrt{[(1\tfrac{1}{2})^2+4]}+1\tfrac{1}{2} \\ &= \sqrt{[2\tfrac{1}{4}+4]}+1\tfrac{1}{2} \\ &= \sqrt{6\tfrac{1}{4}}+1\tfrac{1}{2} \\ &= 2\tfrac{1}{2}+1\tfrac{1}{2}=4. \end{aligned}$$

OF THE SIX PROBLEMS

First Problem

I have divided ten into two portions; I have multiplied the one of the two portions by the other; after this I have multiplied the one of the two by itself, and the product of the multiplication by itself is four times as much as that of one of the portions by the other.[1]

Computation: Suppose one of the portions to be thing, and the other ten minus thing: you multiply thing by ten minus thing; it is ten things minus a square. Then multiply it by four, because the instance states 'four times as much'. The result will be four times the product of one of the parts multiplied by the other. This is forty things minus four squares. After this you multiply thing by thing, that is to say, one of the portions by itself. This is a square which is equal to forty things minus four squares. Reduce it now by the four squares, and add them to the one square. Then the equation is: forty things are equal to five squares; and one square will be equal to eight roots, that is, sixty-four; the root of this is eight, and this is one of the two portions, namely, that which is to be multiplied by itself. The remainder from the ten is two, and that is the other portion. Thus the question leads you to one of the six cases, namely, that of 'squares equal to roots'. Remark this.

Second Problem

I have divided ten into two portions: I have multiplied each of the parts by itself, and afterwards ten by itself: the product of ten by itself is equal to one of the two parts multiplied by itself, and afterwards by two and seven-ninths; or equal to the other multiplied by itself, and afterwards by six and one-fourth.[2]

Computation: Suppose one of the parts to be thing, and the other ten minus thing. You multiply thing by itself, it is a square; then by two and seven-ninths, this makes it two squares and

1. $x^2 = 4x(10-x) = 40x - 4x^2$
   $5x^2 = 40x$
   $x^2 = 8x$
   $x = 8;\ (10-x) = 2.$

2. $10^2 = x^2 \times 2\frac{7}{9}$
   $100 = x^2 \times \frac{25}{9}$
   $\frac{9}{25} \times 100 = x^2$
   $36 = x^2$
   $6 = x.$

seven-ninths of a square. You afterwards multiply ten by ten; it is a hundred, which must be equal to two squares and seven-ninths of a square. Reduce it to one square, through division by nine twenty-fifths;[1] this being its fifth and four-fifths of its fifth, take now also the fifth and four-fifths of the fifth of a hundred; this is thirty-six, which is equal to one square. Take its root, it is six. This is one of the two portions; and accordingly the other is four. This question leads you, therefore, to one of the six cases, namely, 'squares equal to numbers'.

VARIOUS QUESTIONS

In the same manner, if the question be: 'A square, multiply its root by four of its roots, and the product will be three times the square, with a surplus of fifty dirhems.'[2] Computation: You multiply the root by four roots, it is four squares, which are equal to three squares and fifty dirhems. Remove three squares from the four; there remains one square, equal to fifty dirhems. One root of fifty, multiplied by four roots of the same, gives two hundred, which is equal to three times the square, and a residue of fifty dirhems.

If the instance be: 'A square, which when added to twenty dirhems, is equal to twelve of its roots',[3] then the solution is this: You say, one square and twenty dirhems are equal to twelve roots. Halve the roots and multiply them by themselves; this gives thirty-six. Subtract from this the twenty dirhems, extract the root from the remainder, and subtract it from the moiety of the roots, which is six. The remainder is the root of the square: it is two dirhems, and the square is four.

If the instance be: 'To find a square, of which if one-third be added to three dirhems, and the sum be subtracted from the

1. $\frac{9}{25} = \frac{1}{5} \times \frac{4}{5} + \frac{1}{5}$.
2. $4x^2 = 3x^2 + 50$
   $x^2 = 50$.
3. $x^2 + 20 = 12x$
   $x = 6 \pm \sqrt{36 - 20} = 6 \pm 4 = 10$ or 2

square, the remainder multiplied by itself restores the squares';[1] then the computation is this: If you subtract one-third and three dirhems from the square, there remain two-thirds of it less three dirhems. This is the root. Multiply therefore two-thirds of thing less three dirhems by itself. You say two-thirds by two-thirds is four-ninths of a square; and less two-thirds by three dirhems is two roots: and again, two-thirds by three dirhems is two roots; and less three dirhems by less three dirhems is nine dirhems. You have, therefore, four-ninths of a square and nine dirhems less four roots, which are equal to one root. Add the four roots to the one root, then you have five roots, which are equal to four-ninths of a square and nine dirhems. Complete now your square; that is, multiply the four-ninths of a square by two and a fourth, which gives one square; multiply likewise the nine dirhems by two and a quarter; this gives twenty and a quarter; finally, multiply the five roots by two and a quarter; this gives eleven roots and a quarter. You have, therefore, a square and twenty dirhems and a quarter, equal to eleven roots and a quarter. Reduce this according to what I taught you about halving the roots.

MENSURATION

In any circle, the product of its diameter, multiplied by three and one-seventh, will be equal to the periphery. This is the rule generally followed in practical life, though it is not quite exact. The geometricians have two other methods. One of them is, that you multiply the diameter by itself; then by ten, and hereafter take the root of the product; the root will be the periphery. The other method is used by the astronomers among them: it is this, that you multiply the diameter by sixty-two thousand eight

1. $\left[x-\left(\frac{x}{3}+3\right)\right]^2=x$

or $\left[\frac{2x}{3}-3\right]^2=x$

$$\frac{4x^2}{9}+9=5x$$

$$x^2+20\tfrac{1}{4}=11\tfrac{1}{4}x$$

$$x=9, \text{ or } 2\tfrac{1}{4}.$$

hundred and thirty-two and then divide the product by twenty thousand; the quotient is the periphery. Both methods come very nearly to the same effect.[1]

If you divide the periphery by three and one-seventh, the quotient is the diameter.

The area of any circle will be found by multiplying the moiety of the circumference by the moiety of the diameter; since, in every polygon of equal sides and angles, such as triangles, quadrangles, pentagons and so on, the area is found by multiplying the moiety of the circumference by the moiety of the diameter of the middle circle that may be drawn through it.

If you multiply the diameter of any circle by itself, and subtract from the product one-seventh and half one-seventh of the same, then the remainder is equal to the area of the circle. This comes very nearly to the same result with the method given above.[2]

Every part of a circle may be compared to a bow. It must be either exactly equal to half the circumference, or less or greater than it. This may be ascertained by the arrow of the bow. When this becomes equal to the moiety of the chord, then the arc is exactly the moiety of the circumference: is it shorter than the moiety of the chord, then the bow is less than half the circumference; is the arrow longer than half the chord, then the bow comprises more than half the circumference.

*

If some one says: 'There is a triangular piece of land, two of its sides having ten yards each, and the basis twelve; what must be the length of one side of a quadrate situated within such a triangle?' the solution is this. At first you ascertain the height of

1. The three formulas are,

   1st, $3\frac{1}{7}d = p$ i.e. $3{\cdot}1428d$

   2nd, $\sqrt{10d^2} = p$ i.e. $3{\cdot}16227d$

   3rd, $\frac{d \times 62{,}832}{20\,000} = p$ i.e. $3{\cdot}1416d$

2. The area of a circle whose diameter is d is

$$\pi\frac{d^2}{4} = \frac{22}{7\times 4}d^2 = \left(1 - \frac{1}{7} - \frac{1}{2\times 7}\right)d^2.$$

the triangle, by multiplying the moiety of the basis (which is six), by itself, and subtracting the product, which is thirty-six, from one of the two short sides multiplied by itself, which is one hundred; the remainder is sixty-four: take the root from this; it is eight. This is the height of the triangle. Its area is, therefore, forty-eight yards: such being the product of the height multiplied by the moiety of the basis, which is six. Now we assume that one side of the quadrate inquired for is thing. We multiply it by itself; thus it becomes a square, which we keep in mind. We know that there must remain two triangles on the two sides of the quadrate, and one above it. The two triangles on both sides of it are equal to each other: both having the same height and being rectangular. You find their area by multiplying thing by six less half a thing, which gives six things less half a square. This is the area of both the triangles on the two sides of the quadrate together. The area of the upper triangle will be found by multiplying eight less thing, which is the height, by half one thing. The product is four things less half a square. This altogether is equal to the area of the quadrate plus that of the three triangles: or, ten things are equal to forty-eight, which is the area of the great triangle. One thing from this is four yards and four-fifths of a yard; and this is the length of any side of the quadrate. Here is the figure (Figure 17.1):

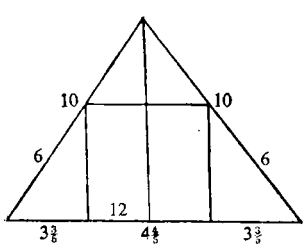


Fig. 17.1.

ON LEGACIES

If he leaves two sons and a daughter,[1] and bequeaths to someone as much as would be the share of a third son, if he had one; then

1. A son is entitled to receive twice as much as a daughter. Were there three sons and one daughter, each son would receive $\frac{2}{7}$ of the residue. Let x be the stranger's legacy.

$$\therefore \tfrac{2}{7}[1-x]=x \quad \therefore x=\tfrac{2}{9}, \quad \text{and } 1-x=\tfrac{7}{9}$$
$$\text{Each son's share} = \tfrac{2}{5}[1-x]=\tfrac{2}{5}\times\tfrac{7}{9}=\tfrac{14}{45}$$
$$\text{The daughter's share} = \tfrac{1}{5}[1-x]\ldots\ldots=\tfrac{7}{45}$$
$$\text{The stranger's legacy} = \tfrac{2}{9}\ldots\ldots\ldots\ldots=\tfrac{10}{45}$$

you must consider, what would be the share of each son, in case he had three. Assume this to be seven, and for the entire heritage take a number, one-fifth of which may be divided into sevenths, and one-seventh of which may be divided into fifths. Such a number is thirty-five. Add to it two-sevenths of the same, namely, ten. This gives forty-five. Herefrom the legatee receives ten, each son fourteen, and the daughter seven.

If he leaves a mother, three sons, and a daughter, and bequeaths to some one as much as the share of one of his sons less the amount of the share of a second daughter, in case he had one; then you distribute the heritage into such a number of parts as may be divided among the actual heirs, and also among the same, if a second daughter were added to them.[1] Such a number is three hundred and thirty-six. The share of the second daughter, if there were one, would be thirty-five, and that of a son eighty; their difference is forty-five, and this is the legacy. Add to it three hundred and thirty-six, the sum is three hundred and eighty-one, which is the number of parts of the entire heritage.

1. Let x be the stranger's legacy; $1-x$ is the residue.

A widow's share of the residue is $\frac{1}{6}$: there remains $\frac{5}{6}[1-x]$, to be distributed among the children.

Since there are 3 sons, and 1 daughter, a son's share is ..... $\frac{2}{7}\times\frac{5}{6}\,|\,1-x\,|$

Were there 3 sons and 2 daughters, a daughter's share would be ..... $\frac{1}{8}\times\frac{5}{6}\,|\,1-x\,|$

The difference $=\frac{9}{56}\times\frac{5}{6}\,|\,1-x\,|$

$\therefore\ x=\frac{45}{336}[1-x]$ $\quad\therefore\ x=\frac{45}{381}$

$1-x=\frac{336}{381}$; the widow's share $=\frac{56}{381}$

the daughter's share $=\frac{40}{381}$

18

BHÁSCARA ÁCHÁRYA

(1114–*c.* 1185)

BHÁSCARA ÁCHÁRYA (Bháscara, the Learned), celebrated Indian astronomer and mathematician, head of the astronomical observatory at Ujjain, where Brahmagupta had served in a similar capacity some 500 years earlier, was born in 1114. He came of an old, learned family, of some nobility and title, possessing an established tradition of scholarship. Bháscara, greatly honoured as one who was deeply conversant with the Vedas, grew to be its outstanding representative. He was widely acclaimed during his own lifetime and later for his extraordinary skill in mechanical arts, as well as in the sciences, such as astronomy and mathematics. Bháscara's grandson, Changadeva, a court astronomer and chief astrologer for King Singhana, was the recipient of a grant of land and the financial means with which he founded a college where the works composed by Bháscara and by Bháscara's ancestors and descendants were taught. An inscription discovered in a ruined temple in the Bombay Presidency extols the virtues and achievements of Bháscara and records the grant as having been made in 1206, on the occasion of a lunar eclipse.

By his own statement, Bháscara completed his great astronomical work the *Sidd'hanta-siromani* in the year 1150 when he was thirty-six years of age. The survival of the original text of Bháscara's *Sidd'hanta* is due to the respect amounting almost to veneration with which the work was regarded by succeeding writers and the method employed by them in conveying Bháscara's teaching. In the numerous commentaries on the *Sidd'hanta-siromani* each passage was stated exactly as it appeared in the original work. This was followed by the commentator's explanation of the passage and such examples as he considered useful as illustrations. H. T. Colebrooke, whose translation (1817) of the mathematical chapters of Bháscara's *Sidd'hanta* remains unsurpassed, found many commentaries written in Sanskrit and one written in Persian. These commentaries, widely separated in time and distance, were in remarkable agreement in their recitals of the original text. The

consistency which thus appeared in works published at different times and in different languages constitutes clear and convincing evidence of the authenticity of the text now in our possession.

Four chapters of the original work have survived. Two are mathematical. Chapter 1, *Lilavati*, deals with arithmetic and geometrical-numerical problems; Chapter 2, *Vija-Ganita*, deals with algebraic problems. Although Bháscara freely acknowledged his dependence on previous writers whose work he compiled, his correction and improvement of their methods was substantial. Moreover, in his own contributions, Bháscara advanced far beyond any who preceded him. Bháscara presented a complete and systematic explanation of the Indian method of solving determinate and indeterminate problems, of the notation employed, of the rules for positive and negative numbers, and the use of colour to represent quantities, both known and unknown. The work also included problems in permutations and combinations, and a chapter on operations with zero in which it is clear that Bháscara knew that there could be no exact division by zero, an infinite quotient resulting from division of any quantity by a number which is made to approach zero as a limit. Bháscara's problems, more varied than those of his predecessors, included the topics of purchase and sale, interest, gems, gold, and permutation of the syllables in a stanza, military marches, excavations, and the measures of grain. However, his indeterminate problems in number theory provided the greatest and most lasting interest. The famous challenge sent by Fermat to Wallis and Brouncker (1657) concerned such a problem. It may be stated as requiring the solution of the equation $cx^2 + 1 = y^2$. As late as 1946, the *Bulletin of the Calcutta Mathematical Society* (v. 38, pp. 21–4), carried an article discussing a problem proposed by Bháscara (Par. 182, *Vija-Ganita*). Numerical solutions of the problems were given in the *Bulletin* in addition to those found by Bháscara and, in line with modern interests, the general formulas for the solution were also presented. The history of the theory of numbers contains ample testimony of continuing interest in Bháscara's problems.

LILAVATI[1]

CHAPTER III

Section VI: Rule of Proportion

74. Rule of three inverse.[2]

If the fruit diminish as the requisition increases, or augment as that decreases, they, who are skilled in accounts, consider the rule of three terms to be inverted.[3]

When there is diminution of fruit, if there be increase of requisition, and increase of fruit if there be diminution of requisition, then the inverse rule of three is [employed]. For instance,

75. When the value of living beings[4] is regulated by their age; and in the case of gold, where the weight and touch are compared; or when heaps[5] are subdivided; let the inverted rule of three terms be [used].

76. Example. If a female slave sixteen years of age, bring thirty-two [*nishcas*], what will one aged twenty cost? If an ox, which has been worked a second year, sell for four *nishcas* what will one, which has been worked six years, cost?

1st Qu. Statement: 16 32 20. Answer: 25-3/5 *nishcas*.

2nd Qu. Statement: 2 4 6. Answer: 1-1/3 *nishca*.

77. Example. If a *gadyánaca* of gold of the touch of ten may be

1. Selections from chapters entitled, *Lilavati* and *Vija-Ganita*, of the *Sidd'hanta-siromani* are taken from the translation from the Sanskrit: *Algebra, with Arithmetic and Mensuration* by Henry Thomas Colebrooke.

2. *Vyasta-trairasica* or *Viloma-trairasica*, rule of three terms inverse.

3. The method of performing the inverse rule has been already taught (§ 70). 'In the inverse method, the operation is reversed.' That is, the fruit to be multiplied by the argument and divided by the demand. – Sur.

When fruit increases or decreases, as the demand is augmented or diminished, the direct rule (*crama-trairasica*) is used. Else the inverse. – Gan.

4. Slaves and cattle. The price of the older is less; of the younger, greater. – Gang. and Sur.

5. When heaps of grain, which had been meted with a small measure, are again meted with a larger one, the number decreases; and when those, which had been meted with a large measure, are again meted with a smaller one, there is increase of number. – Gang. and Sur.

had for one *nishca* [of silver], what weight of gold of fifteen touch may be bought for the same price?

Statement: 10 1 15. Answer 2/3.

78. Example. A heap of grain having been meted with a measure containing seven *ad'hacas*, if a hundred such measures were found, what would be the result with one containing five *ad'hacas*?

Statement: 7 100 5. Answer: 140.

80. Example. If the interest of a hundred for a month be five, say what is the interest of sixteen for a year? Find likewise the time from the principal and interest; and knowing the time and produce, tell the principal sum.

$$\begin{matrix} 1 & 12 \\ 5 & \end{matrix}$$

Statement: 100 16 Answer: the interest is 9-3/5.

$$\begin{matrix} & 1 \\ 5 & 48/5 \end{matrix}$$

To find the time; Statement: 100 16 Answer: months 12.

$$\begin{matrix} 1 & 12 \\ 5 & 48/5 \end{matrix}$$

To find the principal; Statement: 100

Answer: principal 16.

81. Example. If the interest of a hundred for a month and one-third, be five and one-fifth, say what is the interest of sixty-two and a half for three months and one-fifth?

$$\begin{matrix} \frac{4}{3} & \frac{16}{5} \\ \frac{100}{1} & \frac{125}{2} \\ \frac{26}{5} & \end{matrix}$$

Statement: Answer: interest 7-4/5.

82. Example of the rule of seven: If eight, best, variegated, silk scarfs, measuring three cubits in breadth and eight in length, cost a hundred [*nishcas*]; say quickly, merchant, if thou understand trade, what a like scarf, three and a half cubits long and half a cubit wide, will cost.

Statement: 3 1/2
8 7/2
8 1
100

Answer: *Nishca* 0, *drammas* 14, *panas* 9, *cacini* 1, *cowryshells* 6-2/3.

83. Example of the rule of nine: If thirty benches, twelve fingers thick, square of four wide, and fourteen cubits long, cost a hundred [*nishcas*]; tell me, my friend, what price will fourteen benches fetch, which are four less in every dimension?

Statement: 12 8
16 12
14 10
30 14
100

Answer: *Nishcas* 16-2/3.

CHAPTER IV

Section VI: Permutation and Combination

114. Example: In a pleasant, spacious and elegant edifice, with eight doors,[1] constructed by a skilful architect, as a palace for the lord of the land, tell me the permutations of apertures taken one, two, three, etc.[2] Say, mathematician, how many are the combinations in one composition, with ingredients of six different tastes, sweet, pungent, astringent, sour, salt and bitter,[3] taking them by ones, twos or threes, etc.

Statement [1st Example]: 8 7 6 5 4 3 2 1
1 2 3 4 5 6 7 8.

Answer: the number of ways in which the doors may be opened by ones, twos, or threes, etc. is 8, 28, 56, 70, 56, 28, 8, 1. And the
1 2 3 4 5 6 7 8
changes on the apertures of the octagon[4] palace amount to 255.

Statement 2nd Example: 6 5 4 3 2 1
1 2 3 4 5 6.

1. *Muc'ha*, aperture for the admission of air: a door or window; (same with *gavacsha*; – Gan.) a portico or terrace, (*bhumi-visesha*; – Gang. and Sur.).

2. The variations of one window or portico open (or terrace unroofed) and the rest closed; two open, and the rest shut; and so forth.

3. *Amera-cosha* 1.3.18.

4. An octagon building, with eight doors (or windows; porticos or terraces); facing the eight cardinal points of the horizon, is meant. See Gan.

Answer: the number of various preparations with ingredients of divers tastes is 6, 15, 20, 15, 6, 1.[1]
1 2 3 4 5 6

CHAPTER V: PROGRESSIONS

Section I: Arithmetical Progression

115. Rule: Half the period, multiplied by the period added to unity, is the sum of the arithmeticals one, etc. and is named their addition. This, being multiplied by the period added to two, and being divided by three, is the aggregate of the additions.

123. Rule:[2] half a stanza. The sum divided by the period, and the first term subtracted from the quotient, the remainder, divided by half of one less than the number of terms, will be the common difference.

124. Example: On an expedition to seize his enemy's elephants, a king marched two *yojánas* the first day. Say, intelligent calculator, with what increasing rate of daily march did he proceed, since he reached his foe's city, a distance of eighty *yojánas*, in a week?

Statement: First term 2 Com. diff.? Period 7; Sum 80.

Answer: Com. diff. 22/7

125. Rule:[3] From the sum of the progression multiplied by twice the common increase, and added to the square of the difference between the first term and half that increase, the square root being extracted, this root less the first term and added to the [above-mentioned] portion of the increase, being divided by the increase, is pronounced to be the period.

126. Example: A person gave three *drammas* on the first day, and continued to distribute alms increasing by two [a day]; and he thus bestowed on the priests three hundred and sixty *drammas*: say quickly in how many days?

Statement: First term 3; Com. diff. 2; Period? Sum. 360.

Answer: Period 18.

1. Total number of possible combinations, 63. – Gang.

2. The first term, period and sum being known, to find the common difference which is unknown. – Gan.

3. The first term, common difference and sum being known, to find the period which is unknown. – Gan.

Section II: Geometrical Progression

127. Rule: a couplet and a half. The period being an uneven number subtract one, and note 'multiplicator'; being an even one, halve it, and note 'square': until the period be exhausted. Then the produce arising from multiplication and squaring [of the common multiplier] in the inverse order from the last, being lessened by one, the remainder divided by the common multiplier less one, and multiplied by the initial quantity, will be the sum of a progression increasing by a common multiplier.

128. Example: A person gave a mendicant a couple of cowry shells first; and promised a two-fold increase of the alms daily. How many *nishcas* does he give in a month?

Statement: First term, 2. Two-fold increase, 2. Period, 30.

Answer, 2147483646 *cowries*; or 104857 *bishcas*, 9 *drammas*, 9 *panas*, 2 *cacinis*, and 6 *shells*.

CHAPTER VI: PLANE FIGURE

158. Example. Where the difference of the side and upright is seven and hypotenuse is thirteen, say quickly, eminent mathematician, what are the side and upright?[1]

Statement. Difference of side and upright 7. Hypotenuse 13. Proceeding as directed, the side and upright come out 5 and 12. See

159. Rule.[2] The product of two erect bambus being divided by their sum, the quotient is the perpendicular from the junction [intersection] of threads passing reciprocally from the root [of one] to the tip [of the other]. The two bambus, multiplied by an

1. This example of a case where the difference of the sides is given is omitted by Suryadasa, but noticed by Ganesa. Copies of the text vary; some containing, and others omitting, the instance.

2. Having taught fully the method of finding the sides in a right-angled triangle, the author next propounds a special problem. – Gan. To find the perpendicular, the base being unknown. – Sur.

assumed base, and divided by their sum, are the portions of the base on the respective sides of the perpendicular.

160. Example. Tell the perpendicular drawn from the intersection of strings stretched mutually from the roots to the summits of two bambus fifteen and ten cubits high standing upon ground of unknown extent.

Statement: Bambus 15, 10. The perpendicular is found 6.

Next to find the segments of the base: let the ground be assumed 5; the segments come out 3 and 2. Or putting 10, they are 6 and 4. Or taking 15, they are 9 and 6. See the figures (Figure 18.1).

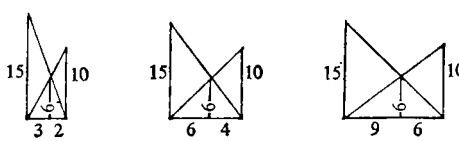


Fig. 18.1.

In every instance the perpendicular is the same: viz. 6.

The proof is in every case by the rule of three: if with a side equal to the base, the bambu be the upright, then with the segment of the base what will be the upright?[1]

161. Aphorism.[2] That figure, though rectilinear, of which sides

1. On each side of the perpendicular, are segments of the base relative to the greater and smaller bambus, and larger or less analogously to them. Hence this proportion. 'If with the sum of the bambus, this sum of the segments equal to the entire base be obtained, then, with the smaller bambu, what is had?' The answer gives the segment, which is relative to the least bambu. Again: 'If with a side equal to the whole base, the higher bambu be the upright, then with a side equal to the segment found as above, what is had?' The answer gives the perpendicular let fall from the intersection of the threads. Here a multiplicator and a divisor equal to the entire base are both cancelled as equal and contrary: and there remain the product of the two bambus for numerator and their sum for denominator. Hence the rule. – Gan.

2. The aphorism explains the nature of impossible figures proposed by dunces. – Sur. It serves as a definition of plane figure (*cshetra*). – Gan. In a triangle or other plane rectilinear figure, one side is always less than the sum of the rest. If equal, the perpendicular is nought, and there is no complete figure. If greater, the sides do not meet. – Sur. Containing no area, it is no figure. – *Caum.* Rang.

are proposed by some presumptuous person, wherein one side[1] exceeds or equals the sum of the other sides, may be known to be no figure.

162. Example: Where sides are proposed two, three, six and twelve in a quadrilateral, or three, six and nine in a triangle, by some presumptuous dunce, know it to be no figure.

CHAPTER XII: PULVERIZER[2]

248-52. Rule. In the first place, as preparatory to the investigation of a pulverizer,[3] the dividend, divisor and additive quantity[4]

1. The principal or greatest side. – Gan. *Caum.* Rang.

2. *Cuttaca-vyavahara* or *Cuttacad'hyaya* determination of a grinding or pulverizing multiplier, or quantity such, that a given number being multiplied by it, and the product added to a given quantity, the sum (or, if the additive be negative, the difference) may be divisible by a given divisor without remainder.

See *Vija-Ganita*, Chapter 2, from which this is borrowed, the contents being copied (with some variation of the order), nearly word for word. For this, as well as the following Chapter 13, on Combination, belongs to algebra rather than arithmetic; according to the remark of the commentator, Ganesa Bhatta: and they are here introduced, as he observes, and treated without employing algebraic forms, to gratify such as are unacquainted with analysis.

The commentator begins by asking 'why this subject has been admitted into a treatise of arithmetic', while a passage of Āryabhata expressly distinguishes it from both arithmetic and algebra: 'the multifarious doctrine of the planets, arithmetic, the pulverizer (*cuttaca*), and analysis (*vija*), and the rest of the science treating of seen objects'; and Brahmagupta, at the beginning of his chapter on Arithmetic, excludes it from this head; when describing the complete mathematician (see Arithm. of Brahm. § 1)? The commentator proceeds to answer, 'Mathematics consist of two branches treating of known and of unknown quantity'; as expressly declared: 'The science of computation (*ganita*) is pronounced two-fold, denominated *vyacta* and *avyacta* (distinct and indistinct).' The investigation of the pulverizer, like the problem of the affected square (*varga-praciti*. See *Vija-ganita*, Ch. 3), is comprehended in algebra, being subservient to its solutions; as hinted by the author. See *Vija.* § 99. The separate mention of the head of investigation of the pulverizer, in passages of Āryabhata and other ancient authors, as well as in those of Bhâscara and the rest ('By arithmetic, by algebra, by investigation of the pulverizer, and by resolution of the affected square, answers are found') is designed as an intimation of the difficulty and importance of the matter; not to indicate it as the subject of a separate treatise: and this, no less than the head of combination treated in the next

are, if practicable, to be reduced by some number.[1] If the number, by which the dividend and divisor are both measured, do not also measure the additive quantity, the question is an ill put [or impossible] one.

249–51. The last remainder, when the dividend and divisor are mutually divided, is their common measure. Being divided by that common measure, they are termed reduced quantities.[2]

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chapter (Chapter 13), with other topics (all exclusive of arithmetic, which comprises logistics and the rest of the enumerated heads terminating with measurement by shadow), falls within algebra, as the precepts of the rules concur with exercise of sagacity to effect the solution. See *Vija*, § 224. It is then true, concludes this commentator, that mathematics consist but of two branches. Nevertheless the subjects of this and of the following chapter are here introduced, to be treated without reference to algebraic solutions, as the *Bhadra-ganita* and other problems have found place in the arithmetical treatises of Narayana and other writers, to be there wrought without algebra; and for the same purpose of gratifying such as are not conversant with this branch. – Gan.

In Brahmagupta's work the whole of algebra is comprised under this title of *Cuttacad'hyaya*, chapter on the pulverizer. See Brahm. Ch. 18, and Chaturveda on Brahmagupta, Chapter 12, § 66.

3. *Cuttaca* or *Cutta*, from *cutt*, to grind or pulverize; (to multiply: all verbs importing tendency to destruction also signifying multiplication. – Gan.).

The term is here employed in a sense independent of its etymology to signify a multiplier such, that a given dividend being multiplied by it, and a given quantity added to (or subtracted from) the product, the sum (or difference) may be measured by a given divisor. – Sur. on *Vij.-gan.* and *Lil.* Rang. on *Vas.*, Gan. on *Lil.*

The derivative import is, however, retained in the present version to distinguish this from multiplier in general; *cuttaca* being restricted to the particular multiplier of the problem in question.

4. *Cshepa*, or *cshepaca*, or *yuti*, additive. From *cship* to cast or throw in, and from *yu* to mix. A quantity superinduced, being either affirmative or negative, and consequently in some examples an additive, in others a subtractive, term.

Visudd'hi, subtractive quantity, contradistinguished from *cshepa* additive when this is restricted to an affirmative one. See § 263.

1. *Apavartana*, abridgement; abbreviation. – Gan. Depression or reduction to least terms; division without remainder: also the number which serves to divide without residue; the common measure, or common divisor of equal division. – Sur.

2. *Drid'ha*, firm: reduced by the common divisor to the least term. The

Divide mutually the reduced dividend and divisor, until unity be the remainder in the dividend. Place the quotients one under the other; and the additive quantity beneath them, and cipher at the bottom.[1] By the penult multiply the number next above it and add the lowest term. Then reject the last and repeat the operation until a pair of numbers be left. The uppermost of these being abraded[2] by the reduced dividend, the remainder is the quotient. The other [or lower-most] being in like manner abraded by the reduced divisor, the remainder is the multiplier.

252. Thus precisely is the operation when the quotients are an even number.[3] But, if they be odd, the numbers as found must be subtracted from their respective abraders, the residues will be the true quotient and multiplier.

253. Example. Say quickly, mathematician, what is that multiplier, by which two hundred and twenty-one being multiplied, and sixty-five added to the product, the sum divided by a hundred and ninety-five becomes exhausted?

Statement: Dividend 221, Divisor 195. Additive 65.

Here the dividend and divisor being mutually divided, the last of the remainders (or divisors) is 13. By this common measure, the dividend, divisor and additive, being reduced to their least terms, are

Divd. 17, Divr. 15. Addve. 5.

The reduced dividend and divisor being divided reciprocaliy,

---

word is applicable to the reduced additive, as well as to the dividend and divisor. Brahmagupta uses *nich'heda* and *nirapavarta* in this sense. – *Brahm.* 18, § 9.

1, *Tashta*, abraded; from *tacsh*, to pare or abrade: divided, but the residue taken, disregarding the quotient: reduced to a residue. – Sur. As it were a residue after repeated subtractions. – Gang.

Tacshana, the abrader; the divisor employed in such operation.

2. *P'hala-valli*, the series of quotients; to be reduced by the operation forthwith directed to only two terms.

3. Even, as 2, 4, 6, etc. – *Manor.*

and the quotients put one under the other, the additive under them, and cipher at the bottom, the series which results is

1
7
5
0

Then multiplying by the penult the number above it and proceeding as directed, the two quantities are obtained

40
35

These being abraded by the reduced dividend and divisor 17 and 15, the quotient and multiplier are obtained 6 and 5. Or, by the subsequent rule (§ 262), adding them to their abraders multiplied by an assumed number, the quotient and multiplier [putting 1] are 23 and 20; or, putting 2, they are 40 and 35; and so forth.[1]

254. Rule: The multiplier is also found by the method of the pulverizer, the additive quantity and dividend, being either reduced by a common measure [or used unreduced.][2] But if the additive and divisor be so reduced, the multiplier found, being multiplied by the common measure, is the true one.

255. Example. If thou be expert in the investigation of such questions, tell me the precise multiplier by which a hundred being multiplied, with ninety added to the product, or subtracted from it, the sum, or the difference, may be divisible by sixty-three without a remainder.

265. Rule for a conjunct pulverizer. If the divisor be the same and the multipliers various; then, making the sum of those multipliers the dividend, and the sum of the remainders a single remainder, and applying the foregoing method of investigation, the precise multiplier so found is denominated a conjunct one.

266. Example. What quantity is it, which multiplied by five, and divided by sixty-three, gives a residue of seven; and the same multiplied by ten and divided by sixty-three, a remainder of fourteen? declare the number.

Here the sum of the multipliers is made the dividend; and the

1. Putting 3, they are 57 and 50. - Sur. and Gang. 2. Gan.

sum of the residues, a subtractive quantity; and the statement is

Dividend 15
Divisor 63 Subtractive 21.

Or reduced to least terms

Dividend 5
Divisor 21 Subtractive 7.

Proceeding as before, the multiplier is found 14.

CHAPTER XIII: COMBINATION[1]

267. Rule. The product of multiplication of the arithmetical series beginning and increasing by unity and continued to the number of places, will be the variations of number with specific figures: that, divided by the number of digits and multiplied by the sum of the figures, being repeated in the places of figures and added together, will be the sum of the permutations.

268. Example. How many variations of number can there be with two and eight? or with three, nine and eight? or with the continued series from two to nine? and tell promptly the several sums of their numbers.

Statement 1st Example: 2. 8.

Here the number of places is 2. The product of the series from 1 to the number of places and increasing by unity, (1, 2). Thus the permutations of number are found 2.

That product 2, multiplied by the sum of the figures, 10 [2 and 8] is 20; and divided by the number of digits 2, is 10. This, repeated in the places of figures [10
10]
and added together, is 110; the sum of the numbers.

Statement 2nd Example: 3, 9. 8.

The arithmetical series is 1. 2. 3.; of which the product is 6; and so many are the variations of number. That, multiplied by the sum 20, is 120; which, divided by the number of digits 3,

1. *Anca-pasa-vyavahara* or *Anca-pasa'd'hyaya*, concatenation of digits: a mutual mixing of the numbers, as it were a rope or halter of numerals: their variations being likened to a coil. See Gan. and Sur.

gives 40; and this, repeated in the places of figures

[40
40
40]

and summed, makes 4440 the sum of the numbers.

Statement 3rd Example: 2.3.4.5.6.7.8.9.

The arithmetical series beginning and increasing by unity is 1.2.3.4.5.6.7.8. The product gives the permutation of numbers 40320. This, multiplied by the sum of the figures 44, is 1774080; and divided by the number of terms 8, is 221760; and the quotient being repeated in the eight places of figures and summed, the total is the sum of the numbers 2463999935360.

269. Example. How many are the variations of form of the god Sambhu by the exchange of his ten attributes held reciprocally in his several hands: namely the rope, the elephant's hook, the serpent, the tabor, the skull, the trident, the bedstead, the dagger, the arrow, and the bow:[1] as those of Hari by the exchange of the mace, the discus, the lotus and the conch?

Statement: Number of places 10.

In the same mode, as above shown, the variations of form are found 3628800.

So the variations of form of Hari are 24.

1. Sambu or Siva is represented with ten arms, and holding in his ten hands the ten weapons or symbols here specified; and, by changing the several attributes from one hand to another, a variation may be effected in the representation of the idol: in the same manner as the image of Hari or Vishnu is varied by the exchange of his four symbols in his four hands. The twenty-four different representations of Vishnu, arising from this diversity in the manner of placing the weapons or attributes in his four hands, are distinguished by as many discriminative titles of the god allotted to those figures in the theogonies or *Puranas*. It does not appear that distinct titles have been in like manner assigned to any part of the more than three millions of varied representations of Siva.

The ten attributes of Siva are, 1st, *pasa*, a rope or chain for binding an elephant; 2nd, *ancusa*, a hook for guiding an elephant; 3rd, a serpent; 4th, *damaru*, a tabor; 5th, a human skull; 6th, a trident; 7th, *c'hatwanga*, a bedstead, or a club in form of the foot of one; 8th, a dagger; 9th, an arrow; 10th, a bow.

270. Rule. The permutations found as before, being divided by the combinations separately computed for as many places as are filled by like digits, will be the variations of numbers; from which the sum of the numbers will be found as before.

271. Example. How many are the numbers with two, two, one and one? and tell me quickly, mathematician, their sum: also with four, eight, five, five and five; if thou be conversant with the rule of permutation of numbers.

Statement 1st Example: 2.2.1.1.

Here the permutations found as before (§ 267) are 24; First, two places are filled by like digits (2.2); and the combinations for that number of places are 2. Next two other places are filled by like digits (1.1.); and the combinations for these places are also 2. Total 4. The permutations as before 24, divided by (4) the two-fold combinations for two pairs of like digits, give 6 for the variations of number: viz. 2211, 2121, 2112, 1212, 1221, 1122. The sum of the numbers is found as before 9999.

Statement 2nd Example: 4.8.5.5.5.

Here the permutations found as before are 120; which, divided by the combinations for three places 6, give the variations 20: viz. 48555, 84555, 54855, 58455, 55485, 55845, 55548, 55584, 45855, 45585, 45558, 85455, 85545, 85554, 54585, 58545, 55458, 55854, 54558, 58554.

The sum of the numbers comes out 1199988.

274. Rule:[1] a stanza and a half. If the sum of the digits be determinate, the arithmetical series of numbers from one less than the sum of the digits, decreasing by unity, and continued to one less than the places, being divided by one and so forth, and the quotients being multiplied together, the product will be equal to the variations of the number.

This rule must be understood to hold good, provided the sum of the digits be less than the number of places added to nine.

A compendium only has been here delivered for fear of prolixity: since the ocean of calculation has no bounds.

1. To find the combinations with indeterminate digits for a definite sum and a specific number of places. – Gan.

275. Example. How many various numbers are there, with digits standing in five places, the sum of which is thirteen? If thou know, declare them.

Here the sum of the digits less one is 12. The decreasing series from this to one less than the number of digits, divided by unity, etc. being exhibited, the statement is $\frac{12}{1}\ \frac{11}{2}\ \frac{10}{3}\ \frac{9}{4}$. The product of their multiplication $[\frac{11880}{24}]$ is equal to the variations of the number, 495.[1]

277. Joy and happiness is indeed ever increasing in this world for those who have *Lilavati* clasped to their throats, decorated as the members are with neat reduction of fractions, multiplication and involution, pure and perfect as are the solutions, and tasteful as is the speech which is exemplified.

VIJA-GANITA

CHAPTER I

Section II: Logistics of Negative and Affirmative Quantities

ADDITION

3. Rule for addition of affirmative and negative quantities: half a stanza. In the addition of two negative or two affirmative[2] quantities, the sum must be taken: but the difference of an affirmative and a negative quantity is their addition.

4. Example. Tell quickly the result of the numbers three and four, negative or affirmative, taken together: that is, affirmative and negative, or both negative or both affirmative, as separate instances: if thou know the addition of affirmative and negative quantities.

1. 91111, 52222, 13333; each five ways. 55111, 22333; each ten ways. 82111, 73111, 64111, 43222, 61222; each twenty ways. 72211, 53311, 44221, 44311; each thirty ways. 63211, 54211, 53221, 43321; each sixty ways. Total four hundred and ninety-five.

2. *Rina* or *cshaya*, minus; literally debt or loss: negative quantity.
D'hana or *swa*, plus; literally wealth or property: affirmative or positive quantity.

The characters, denoting the quantities known and unknown,[1] should be first written to indicate them generally; and those, which become negative, should be then marked with a dot over them.

Statement: 3.4. Adding them, the sum is found 7.

Statement: 3.4̇. Adding them, the sum is 7̇.

Statement: 3.4̇. Taking the difference, the result of addition comes out 1̇.

Statement: 3.4. Taking the difference, the result of addition is 1.

So in other instances, and in fractions likewise.

SUBTRACTION

5. Rule for subtraction of positive and negative quantities: half a stanza. The quantity to be subtracted being affirmative, becomes negative; or, being negative, becomes affirmative: and the addition of the quantities is then made as above directed.

6. Example: half a stanza. Subtracting two from three, affirmative from affirmative, and negative from negative, or the contrary, tell me quickly the result.

Statement: 3.2. The subtrahend, being affirmative, becomes negative; and the result is 1.

Statement: 3.2̇. The negative subtrahend becomes affirmative; and the result is 1̇.

Statement: 3.2̇. The negative subtrahend becomes affirmative; and the result is 5.

Statement: 3.2. The affirmative subtrahend becomes negative; and the result is 5̇.

MULTIPLICATION

7. Rule for multiplication [and division] of positive and negative quantities: half a stanza. The product of two quantities both

1. *Rasi*, quantity, is either *vyacta*, absolute, specifically known (which is termed *rupa*, form, species); or it is *avyacta*, indistinct, unapparent, unknown (*ajnyata*). It may either be a multiple of the arithmetical unit, or a part of it, or the unit itself.

affirmative, is positive.[1] When a positive quantity and a negative one are multiplied together, the product is negative.[2] The same is the case in division.

Section III: Cipher

12. Rule for addition and subtraction of cipher: part of a stanza. In the addition of cipher, or subtraction of it, the quantity,[3] positive or negative, remains the same. But, subtracted from cipher, it is reversed.[4]

13. Example: half a stanza. Say what is the number three, positive, or [the same number] negative, or cipher, added to cipher, or subtracted from it?[5]

Statement: 3.0̇. These, having cipher added to, or subtracted from, them, remain unchanged: 3.0̇.

Statement: 3.0̇ Subtracted from cipher, they become 3.0̇.[6]

1. The sign only of the product is taught. All the operations upon the numbers are the same which were shown in simple arithmetic (*Lila.* § 14–16). – Crishn.

2. Multiplication, as explained by the commentators, is a sort of addition resting on repetition of the multiplicand as many times as is the number of the multiplicator. Now a multiplicator is of two sorts, positive or negative. If it be positive, the repetition of the multiplicand, which is affirmative or negative, will give correspondently an affirmative or negative product. The multiplication then of positive quantities is positive; and that of a negative multiplicand by a positive multiplier is negative: as is plain. The question for disquisition concerns a negative multiplier. It has been before observed that negation is contrariety. A negative multiplier, therefore, is a contrary one: that is, it makes a contrary repetition of the multiplicand. Such being the case, if the multiplicand be positive (the multiplier being negative), the product will be negative; if the multiplicand be negative, the product will be affirmative. In the latter case the multiplication of two negative quantities gives an affirmative product. In the middle instances, either of the two (multiplicator or multiplicand) being positive, and the other negative, the product is negative: as is taught in the text.

3. Whether absolute, expressed by digits, or unknown, denoted by letter, colour, etc., or an irrational and surd root. – Crishn.

4. In both cases of addition, and in the first of subtraction, the absolute number, unknown quantity, or surd, retains its sign, whether positive or negative. In the other case of subtraction, the sign is reversed. – Crishn.

5. Or having cipher added to, or subtracted from, it. – Crishn.

6. Cipher is neither positive nor negative; and it is therefore exhibited with no distinction of sign. No difference arises from the reversing of it; and none is here shown. – Crishn.

14. Rule: (completing the stanza, § 12.) In the multiplication and the rest of the operations[1] of cipher, the product is cipher; and so it is in multiplication by cipher: but a quantity, divided by cipher, becomes a fraction the denominator of which is cipher.[2]

15. Example: half a stanza. Tell me the product of cipher multiplied by two;[3] and the quotient of it divided by three, and of three divided by cipher; and the square of nought; and its root.

Statement: Multiplicator 2. Multiplicand 0. Product 0.

[Statement: Multiplicator 0. Multiplicand 2. Product 0.[4]]

Statement: Dividend 0. Divisor 3. Quotient 0.

Statement: Dividend 3. Divisor 0. Quotient the fraction 3/0.

This fraction, of which the denominator is cipher, is termed an infinite quantity.[5]

1. Multiplication, division, square and square-root. - Sur. and Crishn.

Multiplication and division are each two-fold: viz. multiplication of nought by a quantity; or the multiplication of this by nought: so division of cipher by a quantity; and the division of this by cipher. But square and square-root are each single. - Crishn.

2. The more the multiplicand is diminished, the smaller is the product; and, if it be reduced in the utmost degree, the product is so likewise: now the utmost diminution of a quantity is the same with the reduction of it to nothing: therefore, if the multiplicand be nought, the product is cipher. In like manner, as the multiplier decreases, so does the product; and, if the multiplier be nought, the product is so too. In fact multiplication is repetition: and, if there be nothing to be repeated, what should the multiplicator repeat, however great it be?

So, if the dividend be diminished, the quotient is reduced: and, if the dividend be reduced to nought, the quotient becomes cipher.

As much as the divisor is diminished, so much is the quotient raised. If the divisor be reduced to the utmost, the quotient is to the utmost increased. But, if it can be specified, that the amount of the quotient is so much, it has not been raised to the utmost: for a quantity greater than that can be assigned. The quotient therefore is indefinitely great, and is rightly termed infinite. - Crishn.

3. Or else multiplying two. - Crishn. 4. Crishn.

5. *Ananta-rasi*, infinite quantity. *C'ha-hara*, fraction having cipher for its denominator.

This fraction, indicating an infinite quantity, is unaltered by addition or subtraction of a fine quantity. For, in reducing the quantities to a common denominator, both the numerator and denominator of the finite quantity,

16. In this quantity consisting of that which has cipher for its divisor, there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth.

Statement: 0. Its square 0. Its root 0.

Section IV: Arithmetical Operations of Unknown Quantities

17. 'So much as' and the colours 'black, blue, yellow and red',[1] and others besides these, have been selected by venerable teachers for names of values[2] of unknown quantities, for the purpose of reckoning therewith.[3]

18. Rule for addition and subtraction: Among quantities so designated, the sum or difference of two or more which are alike must be taken: but such as are unlike,[4] are to be separately set forth.

19. Example. Say quickly, friend, what will affirmative one

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being multiplied by cipher, become nought: and a quantity is unaltered by the addition or subtraction of nought. The numerator of the infinite fraction may indeed be varied by the addition or subtraction of a finite quantity, and so it may by that of another infinite fraction: but whether the finite numerator of a fraction, whose denominator is cipher, be more or less, the quotient of its division by cipher is alike infinite. – Crishn.

1. *Yavat-tavat*, correlatives, *quantum, tantum*; *quot, tot*: as many, or as much of the unknown, as this coefficient number. *Yavat* is relative of the unknown; and *tavat* of its coefficient.

The initial syllables of the *Sanscrit* terms enumerated in the text are employed as marks of unknown quantities; viz. *ya, ca, ni, pi, lo* (also *ha, swe, chi*, etc., for green, white, variegated and so forth). Absolute number is denoted by *ru*, initial of *rupa* form, species. The letters of the alphabet are also used (Ch. 6), as likewise the initial syllables of the terms for the particular things (§ 111).

2. *Mana, miti, unmana* or *unmiti*, measure or value. See note on § 130.

3. For the purpose of reckoning with unknown quantities. – Sur. and Crishn.

4. Heterogeneous: as *rupa*, known or absolute number: *yavat-tavat* (so much as) the first unknown quantity, its square, its cube, its biquadrate, and the product of it and another factor; *calaca* (black) the second unknown quantity, its power, and the product of it with factors: *nilaca* (blue) the third unknown, its powers, and so forth. See Crishn.

unknown with one absolute, and affirmative pair unknown less eight absolute, make, if addition of the two sets take place? and what will they make, if the sum be taken inverting the affirmative and negative signs?[1]

Statement: *ya* 1 *ru* 1 / *ya* 2 *ru* 8 — Answer: the sum is *ya* 3 *ru* 7̇.

CHAPTER IV: SIMPLE EQUATION

121. Example: If thou be expert in this computation, declare quickly two numbers, of which the sum and the difference shall severally be squares; and the product of their multiplication, a cube.

Here the two numbers are put *ya v* 4 *ya v* 5; so assumed, that being added or subtracted, the sum or difference may be a square. The product of their multiplication is *ya v v* 20. It is a cube. By making it equal to the cube of ten times the assumed *yavat-tavat*, and depressing the two sides of the equation by the common divisor, cube of *yavat-tavat*, and proceeding as before, the two numbers are found 10000 and 12500.

122. Example: If thou know two numbers, of which the sum of the cubes is a square, and the sum of their squares is a cube, I acknowledge thee an eminent algebraist.

In this instance the two numbers are put *ya v* 1, *ya v* 2. The sum of their cubes is *ya v gh* 9. This of itself is a square as required. Its root is *ya gh* 3.

Is not that quantity the cube of a square, not the square of a cube? No doubt the root of the square of a cube is cube. But how is the root of the cube of a square, a cube? The answer is, the cube of the square is precisely the same with the square of the cube. Hence if squares be raised twice, or four, or six, or eight times, their roots will be so once, twice, thrice, or four times, respectively. It must be so understood in all cases.

Now the sum of the squares of those quantities is *ya v v* 5. It must be a cube. Making it equal to the cube of five times *yavat-tavat*, and depressing the two sides of the equation by the common

1. Inverting the signs of the first set, of the second, or of both. – Crishn.

divisor, cube of *yavat-tavat*, and proceeding as before, the two numbers are found 625 and 1250.

124. Example: If a bambu, measuring thirty-two cubits, and standing upon level ground, be broken in one place by the force of the wind; and the tip of it meet the ground at sixteen cubits; say, mathematician, at how many cubits from the root is it broken?

CHAPTER VI: EQUATIONS INVOLVING MORE THAN ONE UNKNOWN QUANTITY

This is analysis by equation comprising several colours.

In this, the unknown quantities are numerous, two, three or more. For which *yavat-tavat* and the several colours are to be put to represent the values. They have been settled by the ancient teachers of the science:[1] viz. 'so much as' (*yavat-tavat*), black (*calaca*), blue (*nilaca*), yellow (*pitaca*), red (*lohitaca*), green (*haritaca*), white (*swetaca*), variegated (*chitraca*), tawny (*capilaca*), tan-coloured (*pingala*), grey (*d'humraca*), pink (*patalaca*), white (*savalaca*), black (*syamalaca*), another black (*mechaca*), and so forth. Or letters[2] are to be employed; that is the literal characters *c*, etc. as names of the unknown, to prevent the confounding of them.

Here also, the calculator, performing as before directed every operation implied by the conditions of the example, brings out two equal sides, or more sides, of equation. Then comes the application of the rule: 'From one of the two sides of the equation, subtract the first (letter of) colour of the other. Then subtract

1. *Purvacharyaih*, by former teachers. What particular authors are intended is unexplained. Brahmagupta employs names of colours to designate the unknown, without any remark; whence it appears that the use was already familiar.

2. *Varna*, colour or letter: for the word bears both imports. Former writers used it in the one sense, and directed all the unknown quantities after the first to be represented by colours. But the author takes it also in the second acceptation; and directs letters to be employed, instancing the consonants in their alphabetical order. He appears to intend initial syllables. His predecessors, however, likewise made use of initial syllables for algebraic symbols; for instance the marks of square, cube and other powers; and the sign of a surd root: as well as the initials of colours as tokens of unknown quantities.

from that other side the rest of the (letters or) colours, as well as the known quantities. Hence the one side being divided by the residue of the first (letter or) colour, a value of the (letter or) colour which furnishes the divisor is obtained. If there be many such sides, by so treating those that constitute equations, by pairs, other values are found.

CHAPTER VII: VARIETIES OF QUADRATICS

178. Example from ancient authors: The square of the sum of two numbers, added to the cube of their sum, is equal to twice the sum of their cubes. Tell the numbers, mathematician!

The quantities are to be so put by the intelligent algebraist, as that the solution may not run into length. They are accordingly put *ya* 1 *ca* i and *ya* 1 *ca* 1. Their sum is *ya* 2. Its square *ya v* 4. Its cube *ya gh* 8. The square of the sum added to the cube is *ya gh* 8 *ya v* 4. The cubes of the two quantities respectively are *ya gh* 1 *ya v. ca bh* 3 *ca v. ya bh* 3 *ca gh* i cube of the first; and *ya gh* 1 *ya v. ca bh* 3 *ca v. ya bh* 3 *ca gh* 1 cube of the second; and the sum of these is *ya gh* 2 *ca v. ya bh* 6; and doubled, *ya gh* 4 *ca v. ya bh* 12. Statement for equal subtraction:

ya gh 8 *ya v* 4 *ca v. ya bh* 0
ya gh 4 *ya v* 0 *ca v. ya bh* 12

After equal subtraction made, depressing both sides by the common divisor *ya*, and superadding unity, the root of the first side of equation is *ya* 2 *ru* 1. Roots of the other side (*ca v* 12 *ru* 1) are investigated by the rule of the affected square, and are L2 G7 or *L*28 *G*97. 'Least' root is a value of *ca*. Making an equation of a 'greatest' root with *ya* 2 *ru* 1, the value of *ya* is obtained: viz. 3 or 48. Substitution being made with the respective values, the two quantities come out 1 and 5, or 20 and 76, and so forth.

182. Example: Most learned algebraist! tell various pairs of integer numbers, the difference of which is a square, and the sum of their squares a cube.

Put the two numbers *ya* 1 and *ca* 1. Their difference is *ya* i *ca* 1. Making it equal to the square of *ni*, the value of *ya* is had, *ca* 1 *ni v* i. Substituting with this for *ya*, the two quantities become *ca* 1 *ni v* i and *ca* 1. The sum of their squares is *ca v* 2 *ni v*.

ca bh 2 *ni v v*. 1. It is a cube. Make it then equal to the cube of the square of *ni*; and, subtraction taking place, there results, in the first side of equation, *ni v gh* 1 *ni v v* 1; and, in the second, *ca v* 2 *ni v*. *ca bh* 2. Multiplying both sides by two and superadding the biquadrate of *ni*, the square-root of the second side of the equation is *ca* 2 *ni v* 1. Depressing the first side by the biquadrate of *ni* as common divisor, to *ni v* 2 *ru* 1, the roots investigated by the rule of affected square, are L5 G7; or *L*29 *G*41. Then multiplying 'greatest' by square of 'least', conformably to the rule (§ 180), it comes out *G*175, or 34481. 'Least' root is the value of *ni*. Substituting with that, the former root becomes *ca* 2 *ru* 25; or *ca* 2 *ru* 841. Making an equation of this with 'greatest' root, the value of *ca* is obtained 100 or 17661. Substituting these values respectively, the pair of numbers is brought out 75 and 100; or 16820 and 17661; and so forth.

184. Example: Say in what period (or number of terms) is the sum of a progression continued to a certain period tripled; its first term being three and the common difference two?

CHAPTER IX: CONCLUSION

A particle of tuition conveys science to a comprehensive mind; and having reached it, expands of its own impulse. As oil poured upon water, as a secret entrusted to the vile, as alms bestowed upon the worthy, however little, so does science infused into a wise mind spread by intrinsic force.

It is apparent to men of clear understanding, that the rule of three terms constitutes arithmetic; and sagacity, algebra. Accordingly I have said in the chapter on Spherics:

224. 'The rule of three terms is arithmetic; spotless understanding is algebra.'[1] What is there unknown to the intelligent? Therefore, for the dull alone, it[2] is set forth.'

1. *Vija*.

2. The solution of certain problems set forth in the section. The preceding stanza, a part of which is cited by the scholiast of the *Lilavati* (Ch. 12), premises, 'I deliver for the instruction of youth a few answers of problems found by arithmetic, algebra, the pulverizer, the affected square, the sphere, and [astronomical] instruments'. *Gol.* Sect. II. § 2.

225. To augment wisdom and strengthen confidence, read, do read, mathematician, this abridgment elegant in style, easily understood by youth, comprising the whole essence of computation, and containing the demonstration of its principles, replete with excellence and void of defect.

19

LI YEH

(*c.* 1178–*c.*1265)

LI YEH, one of China's greatest mathematicians, was born in Luan-ch'eng in Northern China at a time of prolonged strife between the northern kingdom and the Sung monarchy in the south. He undoubtedly received an excellent education, for we learn that he passed a government examination for civil servants and that having served the Ch'in government in various capacities as a civil officer, he rose to the position of governor of Ch'in Chou. When the province was seized by the Mongol invaders, Li Yeh escaped by fleeing in disguise. Collecting as many books as he could, he retired to a life of seclusion, study and poverty. During the period of the ascendancy of Kublai Khan, Li Yeh was summoned by the conqueror and questioned concerning governmental service, but he assumed no duties under the Khan's government at that time. We hear that for many years he taught ever increasing numbers of pupils in a Mongolian territory. In 1264, Li Yeh declined Kublai's offer of an office on the grounds of his advanced age and infirmity. However, he could not refuse the subsequent insistent offer of a chair in the Han-lin academy. His tenure was brief. After only a few months, Li Yeh resigned and the following year he died at the age of eighty-seven.

Li Yeh was the author of hundreds of manuscripts. The chief mathematical works among them were his *Tse-yuan Hai-ching* ('Sea Mirror of the Circle-measurement') written (1248) in twelve books and his *I-ku Yen-tuan* (1257), a treatise on algebra. Li Yeh presented an algebraic treatment of problems in mensuration, trigonometry and other topics. His general method, *t'ien-yuen-shu* ('the method of the celestial element'), came to be known as *tengen jutsu* in Japanese works. The appearance of a circle for zero in the works of Li Yeh and his contemporaries in China is believed to be due to mathematical treatises brought to China from foreign lands to the west, by numerous visitors, such as the Venetians, Nicolo and Maffeo Polo and Nicolo's famous son, Marco Polo, who came to China during that period. Employing a place

value arrangement of the terms of an equation, Li Yeh solved the equation with the aid of computing rods, red and black for positive and negative quantities. This procedure survived for centuries in both China and Japan. A new notation for negative numbers was introduced by Li Yeh. Replacing the use of red ink to represent positive numbers and black ink for negative numbers, he employed only a diagonal stroke over the representation of the number, or a part of it, to show that it was negative. Li Yeh was also known by his familiar name, Jin-king and by his *nom de plume*, Li Ching Chai.

THE ALGEBRAIC WORKS OF LI YEH

From The Development of Mathematics in China and Japan
by Yoshio Mikami

Li Yeh proceeds in the twelve books or chapters of the 'Sea-Mirror' to establish his rules. In the opening of the treatise he gives a diagram which is as shown in Figure 19.1.

In the first book or chapter Li Yeh describes the relations of

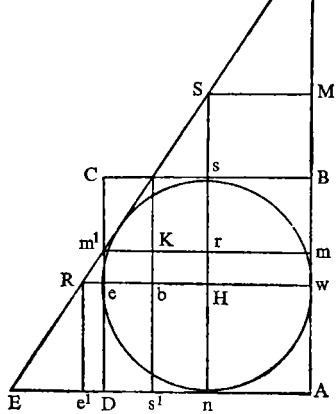


Fig. 19.1.

the various parts in this figure, and the second and further books are devoted to the consideration of various rules of calculation in relation to these quantities. The treatment is done in the algebraical way.

The same author's subsequent composition, the *I-ku Yen-tuan* was also devoted to algebraical treatment. The ways of representing algebraical expressions are however different in these two works.

We have already remarked upon Ch'in Chiu-shao's employment of the term *t'ien-yuen*, which means celestial element. This same term was also employed by Li Yeh. Ch'in referred to the celestial element in his treatment of indeterminate problems, as we have said; he did not however employ the term in connexion with his manipulation of numerical equations, with which he has become so prominent in history. It was in this latter connexion that Li Yeh used the term celestial element.

In the algebraical process of Li Yeh's celestial element in his *I-ku Yen-tuan* unity is employed as the representative of an unknown number. This was certainly done with the aid of a calculating piece. But in writing in his work Li uses the ideogram *yuen* or element written by the side of it to distinguish it from other terms in an expression, which are arranged in a vertical column. The absolute term comes under the linear term and it is marked by the ideogram *tai*, perhaps implying the meaning that it is in the place of the *tai-chi* or extreme limit or great extreme. The square of the celestial element is arranged immediately above the element; next above comes the cube; and so on upwards. Thus the equation

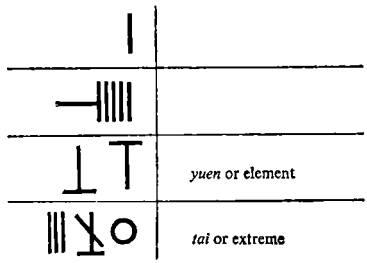
$$x^3+15x^2+66x-360=0$$

would be written as given annexed.

The square and further terms are not expressly indicated, but these are designated by the relative positions to the great extreme or the absolute term. In practice the two ideograms *yuen* and *tai* are not necessary; one of them may be advantageously omitted.

Li Yeh's way of arranging equations as found in the 'Sea-Mirror' is utterly different from that of Ch'in, who had followed the development of the root-extraction process of old times. It is

utterly unknown, however, to us as to what reason or reasons had induced our author to adopt such an arrangement. But in the *I-ku Yen-tuan* Li Yeh changes his arrangement in the same way as Ch'in has done, bringing the absolute term uppermost and other terms successively below it. And this way was destined to prevail thenceforth.



The Chinese had been accustomed long before to use red and black calculating pieces to distinguish positive and negative numbers, and the equations were ultimately to be arranged with these pieces. But it is not convenient to use two different colours in writing and especially in the case of printing. Thus there soon arose the usage of distinguishing numbers of different signs, when representing them in writing, by referring to a specially designed symbol. Such a symbol is met with in both works of Li Yeh. If we are not sure whether he was the first Chinese who used the symbol, yet his works are perhaps the oldest writings wherein it was made use of, that are transmitted to our time. This symbol consists of a diagonal stroke drawn through the right-hand figure. Thus

stand respectively for -10724, -9 and -10200.

20

CH'IN CHIU-SHAO

(*c.* 1250)

CH'IN CHIU-SHAO (Tsin Keu-Chaou), one of the greatest scholars of the thirteenth century, flourished under the Sung Dynasty in China about the middle of the century. A biographical account of his life places him at the age of seventeen as a captain of volunteers in the army, capable of amazing feats of skill in riding, fencing and archery. The intellectual talents which drew him throughout his life to deep learning in astronomy, music, poetry, architecture and mathematics were in evidence from his childhood. Yet it is admitted that while his knowledge and works were respected and admired, his brilliance and superiority in every field made him proud, that moreover, he loved extravagance, had little faith in his friends and did not always live up to the highest ideals of character. From Ch'in's own writing we learn that although he had escaped the hazards of military combat unscathed, ten long years of suffering the ravages of illness brought him to the depths of despair. Not until his faith enabled him to acknowledge and bow to the inevitability of fate was his torment alleviated. Then, as he resumed his studies, he also recovered his vigour. Devoting himself completely to the study of mathematics, Ch'in retired into seclusion where he wrote his most important work, the *Su-shu Chiu-chang* ('Nine Sections of the Art of Numbers' or 'Nine Sections in Mathematics'). When he had fully recovered, Ch'in returned to the service of the Sung government. During the period 1253–8 he served in the region bordering the Yang-tse River. Later he became governor of Ch'iung Chou. His last appointment under Sung was as governor of Mei Choi. Ch'in was serving in this office when he died.

Ch'in Chiu-Shao's *Su-shu Chiu-chang* or *Su-hsiao Chiu-chang* is not a reworking or redevelopment of the old *K'iu-ch'ang Suan-shu* ('Arithmetic in Nine Sections'). The problems in Ch'in's works are grouped in nine sections, but the classifications are not the same as in the older work. The title given to the work by Ch'in is said by some to have been merely *Su-shu* or *Su-hsiao*. The present text of Ch'in's treatise, including

his preface with its autobiographical notes, has been obtained from a copy made by Chang Ch'i-mei in 1646. Chang Ch'i-mei's sources go back to the *Yuan-lo Tai-tien* (Yuan-lo Great Collections). The eighty-one problems of Ch'in's *Su-shu* include both determinate and indeterminate types. In addition to some traditional Chinese problems there are also problems whose sources are astronomy, land surveying, state service, commerce and military science. The *Su-shu* is distinguished for its advance in algebraic techniques. It contains a detailed description of the Chinese method, *Tai-yen ch'iu-i-shu* or Ta-yen ('Great Extension'), for solving indeterminate problems. It also presents the method for solving determinate equations, which was called *Lih-tien-yuen-yih* ('Setting Up the Celestial Monad'). Here, the unknown quantity is taken as the celestial element or monad and the principle of place values for algebraic terms is employed. Many illustrations are given of the Chinese method known as *Ling-lung-kae-fang* ('Harmoniously Alternating Evolution') for finding the roots of numerical equations. Positive and negative numbers are written by Ch'in in red and black ink respectively, corresponding to the red and black bamboo computing rods. Ch'in's procedure in his *Ling-lung-kae-fang* involves place values for algebraic terms and is remarkably like Horner's method of the early nineteenth century.

NINE SECTIONS OF THE ART OF NUMBERS

From Chinese Researches, *by Alexander Wylie*

In examining the productions of the Chinese one finds considerable difficulty in assigning the precise date for the origin of any mathematical process; for on almost every point, where we consult a native author, we find references to some still earlier work on the subject. The high veneration with which it has been customary for them to look upon the labours of the ancients, has made them more desirous of elucidating the works of their predecessors than of seeking fame in an untrodden path; so that some of their most important formulae have reached the state in which we now find them by an almost innumerable series of increments. One of the most remarkable of these is the *Ta-yen*, 'Great Extension', a rule for the resolution of indeterminate

problems. This rule is met with in embryo in Sun Tsze's Arithmetical Classic under the name of *Wuh-puh-chi-soo*, 'Unknown Numerical Quantities', where after a general statement in four lines of rhyme the following question is proposed:

Given an unknown number, which when divided by 3, leaves a remainder of 2; when divided by 5, it leaves 3; and when divided by 7, it leaves 2; what is the number? Answer: 23.

This is followed by a special rule for working out the problem, in terms sufficiently concise and elliptical, to elude the comprehension of the casual reader:

Dividing by 3 with a remainder of 2, set down 140; dividing by 5 with a remainder of 3, set down 63; dividing by 7 with a remainder of 2, set down 30; adding these sums together gives 233, from which subtract 210, and the remainder is the number required.

A more general note succeeds:

For 1 obtained by 3, set down 70; for 1 obtained by 5, set down 21; for 1 obtained by 7, set down 15; when the sum is 106 or above subtract 105 from it, and the remainder is the number required.

In tracing the course of this process we find it gradually becoming clearer till towards the end of the Sung dynasty, when the writings of Tsin Keu-chaou put us in full possession of the principles, and enables us to unravel the meaning of the above mysterious assemblage of numerals. Applying the principles of the *Ta-yen* as there laid down: Multiplying together the three divisors 3, 5 and 7, gives 105 for the *Yen-moo*, 'Extension Parent'. Divide this by the *Ting-moo*, 'Fixed Parent', 7, the quotient 15 is the *Yen-soo*, 'Extension Number'. Divide this again by 7, and there is an overplus of 1, which is the *Ching-suh*, 'Multiplying Term'; by which multiply the Extension Number 15, and the product 15 is the *Yung-soo*, 'Use Number', or as it is given above – *for* 1 *obtained by* 7, *set down* 15. Divide the Extension Parent 105 by the Fixed Parent 5, and the quotient 21 is the Extension Number. Divide this again by 5, and the overplus 1 is the Multiplying Term. Multiply the Extension Number 21 by this, and the product 21 is the Use Number; which is given above – *for* 1 *obtained by* 5, *set down* 21. Divide the Extension Parent 105 by

the Fixed Parent 3, and the quotient 35 is the Extension Number. Divide this again by 3, and there is a *Ke*, 'Remainder' of 2. This Remainder being more than unity is then submitted to a subsidiary process termed *Kiu-yih*, 'Finding Unity', which is the alternate division of the Extension Parent and Remainder by each other, till the remainder is reduced to 1; the result in the present instance is 2, which is the Multiplying Term; by which multiply the Extension Number, and the product 70 is the Use Number, which is the meaning of the sentence – *for* 1 *obtained by* 3, *set down* 70. Having thus obtained the several Use Numbers multiply the corresponding original remainders by these:

$$70\times 2=140;\ 21\times 3=63;\ 15\times 2=30;$$

add these numbers together as stated in the rule, and the sum is 233, from which subtract as many times the Parent Number 105 as it will admit, which making 210 the remainder is 23, the number required.

*

But it is the 'Nine Sections of the Art of Numbers' by Tsin Keu-chaou that we have the most full and explicit details on this subject. Here we have the various applications of this theory worked out at great length; the first problem being to find a solution of a passage in the Yih King treating of the origin of the divining numbers:

Qu. In the Yih King it is said: 'The Great Extension Number is 50, and the Use Number is 49.' Again it is said: 'It is divided into 2, to represent the 2 spheres; 1 is suspended to represent the 3 powers; they are drawn out by 4, to represent the 4 seasons; three changes complete a symbol, and eighteen changes perfect the diagram.' What is the rule for the Extension and what are the several numbers?

The 4 cardinal numbers – 1 the senior male, 2 the junior female, 3 the junior male and 4 the junior female – are used to start with as the Origin Numbers. In a second row, opposite each number respectively, is placed the product of the other three.

Origin Numbers	1	2	3	4
Extension Numbers	24	12	8	6

The sum of these lower numbers gives 50, the *Great Extension*

Number alluded to in the text of the Yih King: 50 being an even number and consequently unsuitable for the Use Number, the Origin Numbers are severally taken in pairs and common divisors sought: where this is obtained the number which gives an odd quotient is divided by it, but the other is not. The result of this last operation gives the Fixed Parent Numbers, and the Extension Numbers are obtained the same as before.

Fixed Parents	1	1	3	4
Extension Numbers	12	12	4	3

Subtract from the respective Extension Numbers as many of the corresponding Fixed Parents as they will admit, so as always to leave some remainder.

Fixed Parents	1	1	3	4
Remainders	1	1	1	3

The three Remainders 1 become the respective Multiplying Terms, and submitting the last Remainder 3 to the process for Finding Unity, the Multiplying Term 3 is obtained. By these four terms multiply the respective Extension Numbers, and the Expansion Use Numbers are obtained.

Fixed Parents	1	1	3	4
Expansion Use	12	12	4	9

The second Origin Number 2 having been reduced 1 at the beginning, once the Extension Parent 12 is consequently added to the second Expansion Number, and the Expansion Numbers then become the Fixed Use Numbers, and stand as follows:

Origin Numbers	1	2	3	4
Fixed Use Numbers ...	12	24	4	9

The four upper numbers here are those employed in drawing the divining straws. The sum of the four lower numbers is 49, which is denominated the *Use Number* in the text of the Yih King, being the number of the straws used in divination. These 49 divining straws being divided at random between the two hands, one hand containing an even number, the other must contain odd. In the present instance the left hand is supposed to hold the odd number 33, though unknown by the enquirer. In drawing them out by 1 at a time, as it is known there must be a

remainder of 1, the number 1 is suspended at first, as it is said in the text, 1 *is suspended to represent the* 3 *powers*. The straws are then drawn in so many successive operations by 2, 3 and 4 at a time, which give respectively the remainders 1, 3 and 1, and these are now put down in three parcels. Making the suspended 1, the first parcel, we have then the following numbers:

Use Numbers	12	24	4	9
Parcels	1	1	3	1

Multiply these parcels by the respective Use Numbers and the Full Numbers are obtained.

Full Numbers	12	24	12	9

Adding together these 4 numbers gives 57, from which subtract the Extension Parent 12 as many times as it will admit, and there is a remainder of 9, which make the dividend, and divide it by 3 the Extension Measure, when the quotient 3 is the junior male, and represents the single stroke symbol.

In this number by changing the odd numbers in the left hand different symbols might be obtained. 1, representing two whole strokes; 2, representing one broken stroke; 3, one whole stroke; and 4, one whole and one broken stroke. Such was the means employed in former times for diving into futurity; the various diagrams of Fo-he being thus obtained were supposed to set forth the destiny of enquirers. Some knowledge of the Yih King is necessary to enable one to understand these diagrams, which are evidently the relics of a very ancient system of divination.

The 2nd problem is to find the epoch of conjunction of the terms of concurrent cycles of time. The question is given as follows:

Let the solar year be equal to $365\frac{1}{4}$ days, the moon's revolution 29-499/940 days, and the *Kea-tsze* 60 days. Suppose in the year A.D. 1246, the 53rd day of the *Kea-tsze* or sexagenary cycle of days is the 1st of the 11th month; the 57th day of the *Kea-tsze* is the winter solstice or 1st day of the solar year; and the 1st day of the *Kea-tsze* is the 9th day of the month. Required the time between two conjunctions of the commencement of these three cycles; also the time that has already elapsed, and how much has yet to run. Answer: The time between two conjunctions,

18,240 years: 225,600 months: 6,662,160 days: number of years already past 9,163: number of years unexpired, 9,077.

The 9th problem is as follows:

A report being raised that 3 rice bins each containing the same amount have been robbed, the original quantity is not known, but it is found that in the left-hand one there is still 1 *ho* left; in the middle one, there is 1 *shing* 4 *ho* left; in the right-hand one there is 1 *ho* remaining; the thieves being caught, A confesses that he took a horse-ladle at night and filled it several times out of the left-hand bin, putting the contents in a bag; B confesses having hastily taken a wooden shoe several times full, out of the middle bin; C says he took a bowl and filled it successively out of the right-hand bin. Examining the three vessels the horse-ladle is found to contain 1 *shing* 9 *ho*, the wooden shoe, 1 *shing* 7 *ho*, and the bowl, 1 *shing* 2 *ho*. What is the amount of rice lost, and how much did each take? Answer: Lost, 9 *shih* 5 *tow* 6 *shing* 3 *ho*. St olen by A, 3 *shih* 1 *tow* 9 *shing* 2 *ho*; B, 3 *shih* 1 *tow* 7 *shing* 9 *ho*; C, 3 *shi* 1 *tow* 9 *shing* 2 *ho*.

At the close of the thirteenth century, we find a grand step made toward the theory of analysis, in the publication of the *Lih-tien-yuen-yih*, 'Setting up the Celestial Monad', a new branch of arithmetic, which may with propriety be termed the Chinese algebra.

In the *Tien-yuen-yih*, unity is employed as the representative of an unknown number; this being combined with an extension of the theory of local value, in order to represent the successive powers of the Monad or unknown number. The Monad has the character *Yuen* written by the side of it, to distinguish its place in the column. Immediately below the Monad is the place for the natural number, which is marked by the character *Tai*, implying that it is the place of *Tai-kieh*, or extreme limit. Immediately above the Monad is the square; next above is the cube; again above, the fourth power, and so on upwards *ad infinitum*. Thus the equation $x^3+15x^2+66x-360=0$ would be written:

1 Cube of Monad.
15 Square of Monad.
66 Monad.
−360 Natural Number.

The cube, square, etc., are all indicated by their relative position with respect to *Tai*, the characters being used only to express coefficients.

One example taken from a great number in Tsin's work, will best illustrate this. It is required to extract the root of

$$-x^4+1534464x^2-526727677600,$$

or to find the value of x.

720	*Shang.* Value of x, the Monad.
−526727577600	*Shih.*
−14940217600	
14940217600	
0	*Fang.*
731124800	
776249600	
747010880	
1534464	*Shang-lien.*
1044464	
64464	
−1405536	
−1461936	
0	*Hia-lien.*
−700	
−1400	
−2100	
−2800	
−2820	
−1	*Yu.*
−1	

In the above example, the several terms are set down with the Chinese names *Shih*, *Fang*, etc., against them, the units in each

being 2 places more to the left than in the term above it, because there is found to be 3 figures in the root. The first figure of the root being 7, is set down at the top. The *Yu* −1 is then multiplied by this 7, and the product added to the *Hia-lien* 0, regard being had to the place in the column, which gives −700; the *Hia-lien* −700 is then multiplied by 7, the product being added to the *Shang-lien* 1534464, still preserving the same place in the column, gives 1044464; again this is multiplied by 7, and the product added to the *Fang* 0, gives 731124800; this multiplied by 7 and the product combined with the *Shih*, leaves −14940217600, etc. The second figure in the root being found to be 2, the same process is repeated, moving all the terms to the right 1, 2, 3, etc. places.

21

YANG HUI

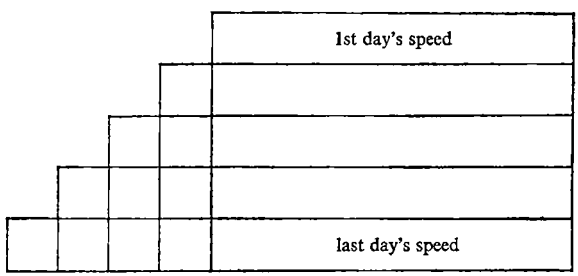
(*c.* 1275)

YANG HUI, the greatest Chinese arithmetician and algebraist of his period, flourished under the Sung Monarchy in China in the latter part of the thirteenth century. A skilful and prolific writer, he was the author of many works, several of which have been preserved to our time. In 1261 Yang Hui wrote his *Hsiang-chieh Chiu-chang Suan-fa* 'The Analysis of the Arithmetical Rules of the Nine Sections', containing an explanation of the last five sections of the old 'Nine Sections' as well as a classified arrangement of the ancient text. This was followed by his 'Supplement to the Analysis' where the computations clearly indicated that in a period of transition in the development of mathematics, when the use of computing rods was still prevalent and reckoning on the suan-pan was rising in favour, Yang Hui was far ahead of his time in his possession of arithmetical methods employing decimal fractions. Five of his other works were grouped together by the celebrated biographer and historian Yuan Yuan (1764–1849) under the title *Song Yang Hoei Soan fa* (or Sung Yang Hui Suan-fa), 'Treatise on Arithmetic by Yang Hoei (Yang Hui) of the Sung Dynasty'. This treatise contained methods for calculating land areas, a geometrical representation of an arithmetic progression and the arithmetical rule for its sum. Yang Hui's *Suan-fa T'ung-pien Pen-mo* gave rules for summing other series including the sum of the squares of the first n natural numbers. In his *Ch'ong-ch'u T'ung-pien Suan-pong*, Yang Hui solved problems with the aid of a division table of a type used in later times with the suan-pan, including among his problems a case involving the solution of a biquadratic equation. Great interest was maintained in Yang Hui's work to a very late date. His *Hsio-ku-tse-ch'i Suan-fa*, 'A Remarkable Arithmetic Adapted from the Ancients', presenting both the ancient rules and the rules of his day, was printed from blocks in the *mi shu* (Statistical Bureau) in 1804.

YANG HUI'S MATHEMATICAL WORKS

From The Development of Mathematics in China and Japan
by Yoshio Mikami

In the 'Analysis of the Nine Sections' Yang Hui illustrates the summation of an arithmetical progression. To find the total distance traversed in fifteen days by a steed who goes 193 Chinese miles (*li*) on the first day and who increases his speed 15 miles every day, Yang takes the product of the number of days and half the sum of the first day's speed and that of the last day, which is found by adding to the former 14 times one day's increase of speed. Here he adds a diagram as shown annexed.



It is obvious that Yang intends to show that the required answer is obtained by taking a quantity corresponding to the area of the rectangle composed by twice this figure, and halving the result.

In the *Suan-fa T'ung-pien Pen-mo* Yang Hui further gives the formulae for the sums of the progressions he calls the triangular and the square, which are

$$1+(1+2)+(1+2+3)+\ldots+(1+2+3+4\ldots+n)=\frac{n(n+1)(n+2)}{6},$$

$$1^2+2^2+3^2+\ldots+n^2=\tfrac{1}{3}n(n+\tfrac{1}{2})(n+1).$$

About these progressions the formulae, or rather rules that lead to these formulae, alone are given; no explanation is tried.

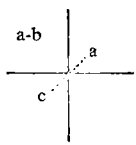
*

Yang Hui considers in the 'Supplement' problems in simple proportion, one of which is this: '*A* runs 100 paces (*a*), while *B* runs 60 paces (*b*). If now *B* starts 100 paces (*c*) ahead of *A*, in how many paces will the latter overtake the former?'

This problem may be obviously solved by the formula of proportion:

$$a:a-b=c:x,$$

which is the very formula Yang Hui employs. He only writes it in the form as annexed and cross multiplies two of the elements and divides the result by the remaining element.



Problems in proportion were considered in the 'Nine Sections' but this scheme of expressing a proportion was given for the first time by Yang Hui, so far as our limited knowledge goes. He calls proportion by the name *hu-huan*, which means literally 'alternate exchange'. He considers in his *Hsu Ku-chai-ch'i Suan-fa* also problems in compound proportion, which he calls *chung hu-huan* or 'doubled alternate exchanges'.

In linear equations Yang considers problems where four or five unknown quantities come in.

In the 'Supplement to the Analysis' Yang considers the problem: 'There is a rectangular field whose breadth is 24 paces and 3·4 feet and whose length is 36 paces and 2·8 feet. What will be its area?' In this problem there is of course nothing that is worthy of notice, but Yang's consideration of it is full of interest. He first treats it expressing the two lengths in terms of the tenth of a foot, but he also tries to carry out the multiplication in this way: When expressed by the decimal parts of the pace, the breadth becomes 24·68 and the length 36·56 paces. If we multiply these together we get

$$24{\cdot}68 \times 36{\cdot}56 = 902{\cdot}3008$$

This shows very well that Yang Hui had been in possession of a way of manipulating directly decimal fractions.

Yang Hui describes in his *Ch'ong-ch'u T'ung-pien Suan-pong* division by the employment of the division-table which is made like the multiplication-table. This table is a part of what has been

employed in later years in the arithmetic with the *suan-pan* or the *soroban* of the Japanese.

In the old 'Nine Sections' we find the problem: 'There is a door whose height is 6·8 feet greater than its breadth and whose diagonal length is 10 feet. What will be the length and breadth?' This is just to find the two sides of a right triangle, whose hypotenuse c and the difference d of two sides a and b are given. To solve it Yang Hui makes use of the relation

$$c^2=2a^2+4\times\left(\frac{d}{2}\right)^2+4\left(\frac{d}{2}\times a\right),$$

which may be easily verified. This is capable of being represented graphically as annexed. If we subtract $2\times(\frac{1}{2}d)^2$ from the whole

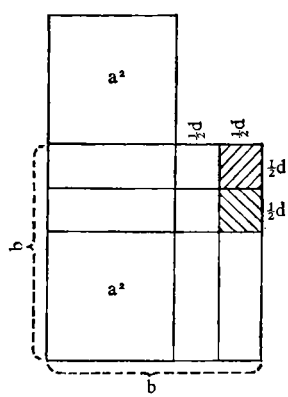


Fig. 21.1.

and take the half of the remainder, the result is the square of $a+\frac{1}{2}d$. We have therefore

$$a=\sqrt{\frac{1}{2}\left\{c^2-2\left(\frac{1}{2}d\right)^2\right\}}-\frac{d}{2}.$$

For this problem Yang Hui tries a second mode of solution. Thus from the relation $c^2=2a^2+2da+d^2$ follows

$$c^2-d^2=2(a^2+da) \quad \text{or} \quad a^2+da=\tfrac{1}{2}(c^2-d^2),$$

to which the method of evolution with an additive term applies,

i.e. the approximate solution of the numerical equation of the second degree can be employed to obtain the value of a.

This method of evolution is explained graphically in the 'Supplement to the Analysis' for the case of a quadratic equation, being referred to as taken from Liu I's *I-ku Kon-yuan*. Yang employs several problems for that purpose and considers several cases separately; but we shall give only one of these cases.

'There is a rectangular field, 864 square paces in area, whose breadth is 12 paces less than the length. It is required to find the breadth.'

Here we have $A=lb=(b+d)b$, where the meanings of the letters employed will be obvious. Consequently $b=$breadth is to be obtained from

$$-864+12x+x^2=0.$$

A method had indeed existed from old times for the solution of such an equation as this, as we have already observed. But Yang Hui explains how such a way is to be followed. Namely, if we know beforehand that $b=24$ and $l=36$ paces, the area of the rectangle will consist of such parts as shown in the left of the figures. Suppose therefore we take $x=20$ and operate as in the case of the extraction of a square root, then we shall see that the area may be divided as in the right-hand figure. If we subtract therefrom 20^2 and 20×12, the remainder will be

$$=2\times(20\times4)+12\times4+4^2.$$

Hence trying to divide roughly the remainder by $(20\times2+12)$, we

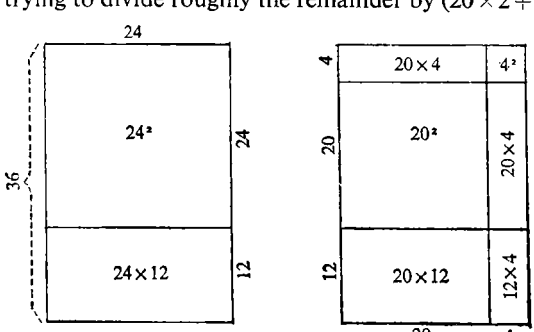


Fig. 21.2.

see that the next figure in the evolved root should be 4. If we now subtract 4^2, there is no remainder for the present case, and we find the breadth to be 24. If however there still is a remainder, the same process may be continued further.

Yang Hui also gives a case of a bi-quadratic equation. In the problem of finding the chord and altitude of a circular segment whose area (a) is 32 and whose diameter (d) is 13 he gives the equation

$$-(2a)^2+4ax^2+4dx^3-5x^4=0,$$

or

$$-4096+128x^2+52x^3-5x^4=0$$

for the determination of the altitude. The solution of this equation is briefly explained and the root is given to be $x=4$.

22

CHU CHI-CHIEH

(*c*. 1300)

CHU CHI-CHIEH (Choo Shi-ki, also Chu Shih-chieh), a Chinese mathematician of remarkable talent, whose works formed a link between the mathematics of China and Japan, was a teacher who wandered over many areas of China earning his livelihood by giving public lessons in mathematics to pupils who gathered about him. He was the author of two great mathematical treatises, one written in 1299 and the other in 1303. The first, *Suan-hsiao Chi-meng* (Introduction to Mathematical Studies) was a compendium of two hundred and fifty-nine problems with their detailed solutions. This work with its skilful employment of the method of the celestial element or monad, explicit rules of operations with positive and negative numbers, and an improved table for use with the abacus, brought Chinese algebra of this period to the high point of its power. The *Suan-hsiao Chi-meng* was reprinted in Korea in 1660 and an edition by the great Japanese mathematician Takebe containing his commentaries was published in 1690. Chu Chi-chieh's treatise was thus directly connected with the advance of mathematics in Japan where his methods were studied and developed by Japanese mathematicians in the seventeenth century and later.

In the second of Chu Chi-chieh's treatises, the *Sze-yuen yuh-kien* ('Precious Mirror of the Four Origins', or 'Precious Mirror of the Four Elements'), he tabulated in triangular form the binomial coefficients as far as the eighth power. The *Precious Mirror* contained two hundred and eighty-eight problems, some requiring an elaborate four monad process and others involving the extraction of roots. Chu Chi-chieh's achievements are all the more striking when it is remembered that his operations were carried out entirely by the use of calculating rods or pieces.

PRECIOUS MIRROR OF THE FOUR ORIGINS

From Chinese Researches *by Alexander Wylie*

At the beginning of the work of Choo Shi-ki, the following table of the 'Ratio of the *Liens* in the involution of numbers, up to the 8th power', is given; not as anything new, however, for it is denominated the '*ancient method*'; but at what time it was first discovered, we have no sufficient date to determine:

						1										Original sum
					1		1									Factors
				1		2		1								Square
			1		3		3		1							Cube
		1		4		6		4		1						Biquadrate
	1		5		10		10		5		1					5th Power
1		6		15		20		15		6		1				6th Power
	7		21		35		35		21		7		1			7th Power
1		8		28		56		70		56		28		8		1 8th Power

Choo Shi-ki published his *Sze-yuen-yuh-kien*, 'Precious Mirror of the Four Origins', about the year 1303; in which he makes a decided step in advance of his contemporaries, by using the Monad for known as well as unknown quantities. Four several Monads are used, which are styled respectively, Heaven, Earth, Man and Thing. The three first are generally (not necessarily) employed to designate known quantities, and the last, *Wuh*, 'Thing'[1] for the unknown quantity. Here again we must yield precedence to the Chinese, for until the time of Vieta, who lived in the sixteenth century, arbitrary representations of numbers were never used in Europe for anything but quantities sought. The *Tien-yuen*, or Heaven, is written below *Tai*, the powers increasing as they descend; Earth is written on the left side of *Tai*, the powers increasing as they recede from *Tai* in a lateral direction; Man is written on the right side, the powers increasing

1. 'The Persians and Arabs employed the word answering to "thing" in their language for the unknown quantity, and the Italians adopted the word "*cosa*" for the same purpose: hence algebra came to be called the *Regola de la Cosa*, in Italy, and the *Cossike Art* in England.' *Penny Cyclopaedia*, vol. 1, p. 325. The coincidence here is curious.

towards the right; and Thing is written above, the powers increasing as they ascend.

Thus the four simple Monads or $a+b+c+x$ are written in the following form:[1]

$$\begin{array}{ccc} & 1 & \\ 1 & * & 1 \\ & 1 & \end{array}$$

The square of the above, or $a^2+2ab+2ac+2ax+b^2+2bc+2bx+c^2+2cx+x^2$, is as follows:

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & 0 & 2 & \\ & & 2 & & \\ 1 & 0 & * & 0 & 1 \\ & & 2 & & \\ & 2 & 0 & 2 & \\ & & 1 & & \end{array}$$

A very little practice is sufficient to render any one perfectly familiar with the several terms of this algorithm. The left being the side for the Earth Monad or b, the 1 removed 2 stages from *Tai* is b^2; in the next column, the upper 2 standing opposite the place for the Thing Monad, signifies twice the product of Earth and Thing, or $2bx$; the lower 2 standing opposite the place for the Heaven Monad, signifies twice the product of Heaven and Earth, or $2ab$; the 2 standing diagonally against *Tai*, indicates twice the product of Earth and Man, the 2 Monads in the same horizontal line, or $2bc$; the other diagonal 2 above *Tai* is twice the product of Heaven and Thing, or $2ax$, etc. The operations connected with this formula are as simple as in the *Tien-yuen*, except it be with regard to multiplication, and even in that, the apparent intricacy is more imaginary than real, the several columns being taken separately and the products added together afterwards. The process of amalgamation and reduction is carried on till all the columns are gradually reduced to one, when the root is extracted the same as before.

*

1. Ed. note: An asterisk occupies the position of the original character *Tai*.

CHU CHI-CHIEH

From The Development of Mathematics in China and Japan
by Yoshio Mikami

Some of the problems solved by the method of the four elements are:

'There is a right triangle whose area is 30 square paces. Given the sum of the two sides to be 17 paces, it is required to find the sum of the first side and the hypotenuse.'

Here the sum of the first side and the hypotenuse is represented by the heaven element and the first side by the earth element. 'These operations being carried out for the heaven and earth elements,' the author writes, 'the final equation will be found to be

$$-3600-3706x-71x^2+34x^3-x^4=0.'$$

'There is a right triangle whose area is 30 square paces. Given the sum of the two sides to be 17 paces, it is required to find the sum of the second side and the hypotenuse.'

For this problem the rule is given thus: 'Take the heaven element for the sum of the second side and the hypotenuse; take the earth element for the second side. These operations being carried out for the heaven and earth elements, the final equation will be obtained to be

$$3600+3706x+71x^2-34x^3+x^4=0.'$$

'There is a right triangle, of which the product of the second side and the sum of the ''five differences'' is equal to the sum of the hypotenuse square and the product of the first side and the hypotenuse; and the quotient of the sum of the ''five sums'' by the first side is equal to the difference of the second side square and the difference of the first side and the hypotenuse. It is required to find the ''yellow magnitude'' plus the three sides.'

Here an explanation is needful. If we designate the three sides of the right triangle by a, b, c, where the last represents the hypotenuse, the 'yellow magnitude' spoken of in the above is $a+b-c$, and the 'five sums' and the 'five differences' are

$$a+b,\ a+c,\ b+c,\ a+(a+b),\ b+(a+b);$$
$$b-a,\ c-a,\ c-b,\ c-(b-a),\ (a+b)-c.$$

Chu's consideration of this problem is only too brief, but we

shall reproduce what he says, because it is of the deepest interest in the history of the Chinese mathematics. Says he:

'Take the heaven element for the first side, the earth element for the second side, the man element for the hypotenuse, and the thing element for the quantity sought for. And operating with these four elements we get the equations:[1]

$$\begin{vmatrix} -2 & * & 1 \\ 0 & 1 & 0 \end{vmatrix}, \quad (a)$$

$$\begin{vmatrix} 0 & 4 & * & 4 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, \quad (b)$$

$$\begin{vmatrix} 1 & 0 & * & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix}, \quad (c)$$

$$\begin{vmatrix} 0 & -1 & 0 \\ 2 & * & 0 \\ 0 & 2 & 0 \end{vmatrix}. \quad (d)$$

'Operating with these four equations so as to effect elimination, and exchanging the thing element with the heaven element we get the equations (α) and (β),[2] of which (β) will be taken for the left column. From these we get the equation (γ),[3] which we take to the right column. From the two inside columns (when (β) and (γ) are arranged side by side on the left and right), and from the two outside columns, by multiplication we get the expressions

$$\begin{vmatrix} 2 & -8 & 28 & * \\ 0 & -1 & 6 & -2 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \quad (\alpha)$$

$$\begin{vmatrix} -7 & * \\ 0 & 2 \end{vmatrix}, \quad (\beta)$$

$$\begin{vmatrix} 0 & 294 \\ 8 & 3 \\ 0 & -4 \end{vmatrix} \quad (\gamma)$$

1. If we denote the four elements by x, y, z, u, these equations will be expressed by

(a) $x-2y+z=0$,
(b) $2x-x^2+4y-xy^2+4z+xz=0$,
(c) $x^2+y^2-z^2=0$,
(d) $2x+2y-u=0$.

2. (α) $-2x-x^2+28y+6xy-8y^2-xy^2+2y^3=0$,
(β) $2x-7y=0$.

3. $294+3x-4x^2+8xy=0$.

(δ) and (ε),[1] and subtracting these from each other and dividing the result by 3, we obtain the equation (ζ).[2] Evolving it according to the square root-extraction, we obtain $x=14$ paces, which is the required answer.'

(δ) $\begin{vmatrix} 0 \\ 0 \\ 16 \end{vmatrix}$, ($\varepsilon$) $\begin{vmatrix} -2058 \\ -21 \\ 28 \end{vmatrix}$, ($\zeta$) $\begin{vmatrix} -686 \\ -7 \\ 4 \end{vmatrix}$

SUAN-HSIAO CHI-MENG

(*Introduction to Mathematical Studies*)
From A History of Japanese Mathematics
by David Eugene Smith and Yoshio Mikami

Chu Chi-chieh also gives, in the *Suan-hsiao Chi-meng*, rules for the treatment of negative numbers. The following translations are as literal as the circumstances allow:

'When the same-named diminish each other, the different-named should be added together.[3] If then there is no opponent for a positive term, make it negative; and for a negative, make it positive.'[4]

'When the different-named diminish each other the same-named should be added together. If then there is no opponent for a positive, make it positive; and for a negative, make it negative.'[5]

'When the same-named are multiplied together, the product is made positive. When the different-named are multiplied together, the product is made negative.'

1. (δ) $16x^2$, (ε) $-2058-21x+28x^2$. Here of course the factor y is understood as rejected.
2. $-686-7x+4x^2=0$.
3. This is intended to mean that when $(+4)-(+3)=+(4-3)$, then $(+4)-(-3)$ should be $+4+3$.
4. That is, $0-(+4)=-4$, and $0-(-4)=+4$.
5. When $(+p)-(-q)=+p+q$, then $(-p)-(+q)=-(p+q)$. Also, $0+(+4)=+4$, and $0+(-4)=-4$.

23

IMMANUEL BEN JACOB BONFILS

(*c.* 1350)

THE physician, mathematician and astronomer, Rabbi Immanuel Bonfils, taught and wrote in Orange, France, and also in Tarascon, during the fourteenth century. Manuscripts of fifteen of his works have survived to our time. Nine of them are in the National Library of Paris. The astronomical observations which were made and published by Bonfils primarily as an aid in the determination of the Hebrew calendar, enjoyed great popularity in his day. Many manuscript copies of this work including extensions to the end of the fifteenth century are extant. Immanuel's highly regarded astronomical tables, *Kanfe Nesharim* (Wings of Eagles), earned him the nickname 'Ba'al kenafayim' (Master of the Wings). The tables were translated into Latin in 1406 and commented upon in Greek.

Immanuel Bonfils also taught mathematics and a mathematical manuscript written by him in the Hebrew language, recently rediscovered, shows that as early as the middle of the fourteenth century Bonfils taught arithmetic employing a decimal system of his own devising, with rules of operation forming an exponential calculus. The manuscript entitled *Derekh Hilluq* had been listed by an old Hebrew copyist as 'A method of division by Rabbi Immanuel, and other topics'. This title has now been proved to be a misnomer. The subject-matter of *Derekh Hilluq* is not division but the exposition of an arithmetic which consistently employs the decimal system of notation not only for integers but also for fractions. In 1935, through a brilliant piece of research, the error of this misnomer was discovered by Sarton, Editor of *Isis*. At his invitation, the task of translating and annotating the manuscript was undertaken by Solomon Gandz. Immanuel's short manuscript was divided by Gandz into six paragraphs. The first three paragraphs dealt with Bonfils's new theory of decimal fractions. The remaining paragraphs, unrelated to the first three, proved to be disconnected fragments of a chapter on the extraction of roots belonging to another work by Bonfils (MS. Hebrew 1081, National Library of Paris).

Derekh Hilluq contains no numerical examples. Bonfils, using no symbols whatever and writing in a fashion common to medieval mathematicians, states his step-by-step rules for operating, rather than the underlying laws or principles. It is clear, however, that the theory of positive and negative exponents was well known to him. While the proposals made by Bonfils concerning a decimal system lay concealed for centuries in a misnamed manuscript, they were in their time of no little importance, enunciated and taught as they were, in a vital centre of commerce and culture, the city of Tarascon, by a man whose works were widely known. Immanuel of Tarascon thus takes his place, as Sarton predicted he would, in the line of the great contributors to the development of the decimal system.

A METHOD OF DIVISION BY RABBI IMMANUEL AND OTHER TOPICS

From Invention of the Decimal Fractions *by Solomon Gandz*

INTRODUCTION[1]

1. Terminology and Definitions

Know that the unit is divided into ten parts which are called Primes, and each Prime is divided into ten parts which are called Seconds, and so on into infinity. I also want to call to your attention that I am calling the degrees of the tens Prime Integers, and the hundreds Second Integers, and so on into infinity. The degree of the units, however, I am calling by their name Units, for it is an intermediate between the integers and the fractions. Therefore, if one multiplies units with units the result is units. Furthermore, I am calling the degrees with a higher denominator[2] 'Of a higher denominator'. I mean by that: I am calling the

1. This heading is found in the Hebrew MS., and may have been applied by Bonfils to the first paragraph containing the terminology and definitions, while paragraphs 2 and 3 were regarded as the text proper. It is possible, however, that all the three paragraphs were intended to form the introduction to a larger text, which became lost or was never written.

2. Literally *shem*, 'name'.

Thirds, of a higher denominator than the Seconds, for the Thirds are derived from three, while the Seconds come from two. Similarly, the Fourths are of a higher denominator than the Thirds, and the Fifths than the Fourths. And this applies to the integers as well as to the fractions.

Furthermore, when I say: add this denominator to that denominator, or subtract this denominator from that denomin-

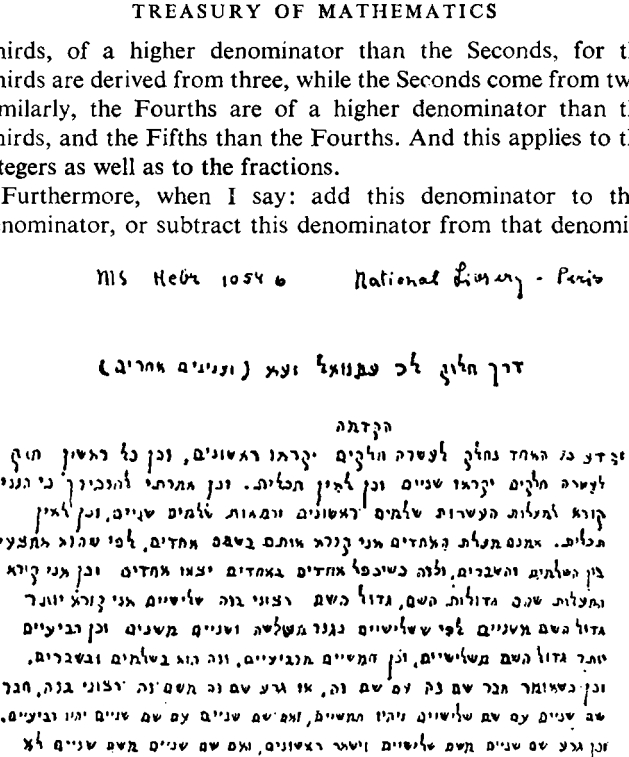


Fig. 23.1. The Hebrew text

ator, I mean by that: add the denominator of the Seconds to the denominator of the Thirds, so that the result will be Fifths, or, the denominator of the Seconds to the denominator of the Seconds, so that there will result Fourths. Or also, subtract the denominator of the Seconds from the denominator of the Thirds and there will remain Primes, or, the denominator of the Seconds from the denominator of the Seconds and nothing will remain, hence it[1] will fall into the degree of the units. And this applies to the integers as well as to the fractions. And if you subtract a large denominator from a small denominator, as when we say: let us subtract Fourths from Seconds, [be it among integers][2] or among fractions, then it[3] will come into the degree of the Seconds of the opposite direction. For instance, when we say: let us subtract the denominator of the Fourths, among the fractions, from the denominator of the Seconds, also among the fractions, then it falls into the degree of the Second integers. Similarly, if we say the same with regard to the integers, i.e.: if we want to subtract the denominator of the Fourth integers from the denominator of the Second integers, then it falls into the Second fractions.[4]

2. Multiplication

If you multiply one number with another number, both being integers or both fractions, add the denominator of the degrees of the one [to the other one], and there the product will be: among the integers, if both are integers, and among the fractions, if both are fractions. If, however, [the one] is an integer and the other one a fraction: if they have the same denominator, then the product will fall into the degree of the units. But if the one has a higher denominator than the other one, then subtract the smaller from the higher one, and where the number of the remaining denominator is, there the product falls; among the integers, if the

1. The Quotient.
2. Added by conjecture.
3. The Quotient.
4. Bonfils omits to say how one shall proceed when the denominator of the fraction is to be deducted from the denominator of the integers, or, when an integer is divided by a fraction. But the rule is given below, at the end of § 3, where it is expressed by a somewhat different formula.

denominator of the integers is higher, or among the fractions, if the denominator of the fractions is higher.

3. Division

If you divide one number by another number, both being integers or both fractions, and the denominator of their degree is the same, then the quotient will fall into the degree of the units. For if you subtract the one denominator from the other denominator, nothing remains; hence it falls into the degree of the units. If [the denominator of] the upper[1] is higher than the denominator of the lower one, subtract the denominator of the lower from the denominator of the upper one, and where the number of the remaining denominator is, there the quotient will fall, in the same direction, i.e. among the integers, if it is an integer, or among the fractions, if it is a fraction. If, however, the lower is higher, then subtract the denominator of the upper from the denominator of the lower one, and where the number of the remaining denominator is, there the quotient will fall, in the opposite direction, i.e. among the fractions, if both are integers, or among the integers, if both are fractions. If, however, the one is an integer and the second is a fraction, may their degrees have the same denominator or different denominators then add the denominators of the degrees, and the quotient will fall where the number of the resulting denominator is: among the fractions, if the upper is a fraction, or among the integers, if the upper is an integer.

TERMINOLOGY

'Elyon, the upper (number, or row), the dividend.
Gera', deduct.
Habber, add.
Haluqqah, the quotient.
Hattekh, melt, reduce a number into the lowest kind of fraction.
Ma'alah, degree; meaning the decimal powers, and also the integers in the sexagesimal fractions.

1. The upper number, or row, is the dividend; the lower is the divisor.

Madregah, degree, of the decimal powers and of the sexagesimal fractions.

Min ha-sheber, the kind of the fraction.

Nifrad, odd.

Rishonim, *Sheniyyim*, Primes, Seconds, of the decimal powers and of the sexagesimal fractions.

Shabber, break, reduce a number into the lowest kind of fractions.

Shebarim, fractions.

Shelemim, integers.

Shem, name, denominator. *Gedol ha-shem*, of a higher denominator.

Shoresh, root.

Tahton, the lower (number, or row), the divisor.

Tur, row. The upper row is the dividend, and the lower row is the divisor.

GREEK GEOMETRY

24

HIPPOCRATES OF CHIOS

(Fifth Century B.C.)

HIPPOCRATES OF CHIOS, author of the first systematic treatise in geometry, was a Greek merchant and mathematician whose historically important activities fell in the third quarter of the fifth century B.C. He is not to be confused with his namesake and younger contemporary, Hippocrates of Cos, ancient Greek physician called the 'Father of Medicine', who flourished in the fourth quarter of the century. Hippocrates of Chios is credited with taking a large part in breaking the secrecy surrounding the mathematics of the Pythagoreans. The circumstances under which Hippocrates of Chios lost his fortune and then perforce resorted to teaching geometry for his livelihood is variously told by ancient writers. Aristotle held him up as an example of a person who may possess considerable capacity in one respect and at the same time be completely inept in others. Hippocrates of Chios, Aristotle said, though skilled in geometry, appeared in other respects to have been weak and stupid, since through his simplicity he had been defrauded of a large sum of money by the collectors of customs in Byzantium. Johannes Philoponus believed that Hippocrates had lost everything to pirates at sea, that he went to Athens to prosecute them, that inordinately protracted legal procedures prolonged his stay there, and that during this time he frequented the schools of the philosophers, becoming extraordinarily skilful in geometry. Referring to Hippocrates as 'one of the Pythagoreans', Iamblichus stated that on account of the misfortune of having lost his property, he was permitted by the Pythagoreans to make money by teaching geometry. This was the means, Iamblichus believed, by which the secrets of the Pythagorean science of geometry were made known and the way opened for the advancement of mathematics.

Thus engaged in teaching geometry, Hippocrates of Chios composed a text book called *Elements*, for those desirous of studying the subject systematically. It was the first treatise of its kind ever written. Fragments of this earliest Greek geometry in our possession have been obtained

from extracts of the history of geometry written by Eudemus of Rhodes, a pupil of Aristotle. Although the actual work by Eudemus is lost, many parts of it are well known to us through quotations from it by later writers. Eudemus's account clearly shows that Hippocrates of Chios was a great originator of mathematics as well as a capable systematizer. He is credited with the discovery of methods of proof which soon became well established methods of procedures in Greek geometry. Hippocrates was the inventor of the method of 'reduction', whereby a problem is solved by showing its dependence upon a subsidiary problem, the solution of the original problem following immediately upon the solution of the subsidiary problem. In the *Elements* written by Hippocrates, this frequently took the form of *reductio ad absurdum*. Ignoring the restriction to the use of the straight line and the circle in the construction of his geometrical figures, Hippocrates solved the Delian problem (required by the oracle of Delos: to construct a cube whose volume is twice that of a given cube) to the subsidiary problem of inserting two mean proportionals between a given line segment and another twice as long. Above all else, Hippocrates of Chios achieved his greatest fame for his highly significant studies and discoveries in the squaring of a lune, the crescent-shaped plane figure bounded by the arcs of two intersecting circles, and for the numerous geometrical theorems he invented and proved.

THE ELEMENTS OF HIPPOCRATES

From Greek Geometry from Thales to Euclid
by George J. Allman

Simplicius has preserved in his *Commentary on the Physics of Aristotle* a pretty full and partly literal extract from the *History of Geometry* of Eudemus, which contains an account of the work of Hippocrates.

SQUARING A LUNE

Let a semicircle $\alpha\beta\gamma$ be described on the straight line $\alpha\beta$; bisect $\alpha\beta$ in δ; from this point δ draw a perpendicular $\delta\gamma$ to $\alpha\beta$, and join $\alpha\gamma$; this will be the side of the square inscribed in the circle of

which $\alpha\beta\gamma$ is the semicircle. On $\alpha\gamma$ describe the semicircle $\alpha\varepsilon\gamma$. Now since the square on $\alpha\beta$ is equal to double the square on $\alpha\gamma$ (and since the squares on the diameters are to each other as the respective circles or semicircles), the semicircle $\alpha\gamma\beta$ is double the semicircle $\alpha\varepsilon\gamma$. The quadrant $\alpha\gamma\delta$ is, therefore, equal to the semicircle $\alpha\varepsilon\gamma$. Take away the common segment lying between the circumference $\alpha\gamma$ and the side of the square; then the remaining lune $\alpha\varepsilon\gamma$ will be equal to the triangle $\alpha\gamma\delta$; but this triangle

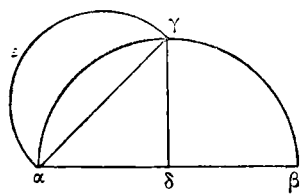


Fig. 24.1.

is equal to a square.

Eudemus,[1] however, tells us in his *History of Geometry*, that Hippocrates demonstrated the quadrature of the lune, not merely the lune on the side of the square, but generally, if one might say so: if, namely, the exterior arc of the lune be equal to a semicircle, or greater or less than it. I shall now put down literally[2] what Eudemus relates, adding only a short explanation by referring to Euclid's *Elements*, on account of the summary manner of Eudemus, who, according to archaic custom, gives concise proofs.

In the second book of his *History of Geometry*, Eudemus says:

'The squaring of lunes seeming to relate to an uncommon class of figures was, on account of their relation to the circle, first treated of by Hippocrates, and was rightly viewed in that connexion. We may, therefore, more fully touch upon and discuss them. He started with and laid down as the first thing useful for them, that similar segments of circles have the same ratio as the squares on their bases. This he proved by showing that circles have the same ratio as the squares on their diameters. Now, as

1. Bretsch., *Geom. vor. Eukl.*, p. 109.

2. Simplicius did not adhere to his intention, or else some transcriber has added to the text.

circles are to each other, so are also similar segments; but similar segments are those which contain the same part of their respective circles, as a semicircle to a semicircle, the third part of a circle to the third part of another circle. For which reason, also, similar segments contain equal angles. The latter are in all semicircles right, in larger segments less than right angles, and so much less as the segments are larger than semicircles; and in smaller segments they are larger than right angles, and so much larger as the segments are smaller than semicircles. Having first shown this, he described a lune which had a semicircle for boundary, by circumscribing a semicircle about a right-angled isosceles triangle and describing on the hypotenuse a segment of a circle similar to those cut off by the sides. The segment over the hypotenuse

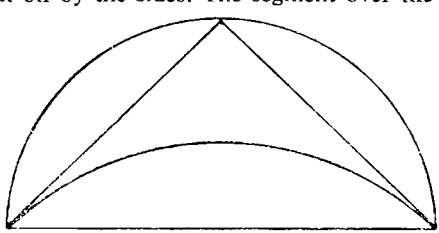


Fig. 24.2.

then being equal to the sum of those on the two other sides, if the common part of the triangle which lies over the segment on the base be added to both, the lune will be equal to the triangle. Since the lune, then, has been shown to be equal to a triangle, it can be squared. Thus, then, Hippocrates, by taking for the exterior arc of the lune that of a semicircle, readily squares the lune.

'Hippocrates next proceeds to square a lune whose exterior arc is greater than a semicircle. In order to do so, he constructs a trapezium[1] having three sides equal to each other, and the fourth – the greater of the two parallel sides – such that the square on it is equal to three times that on any other side; he circumscribes a circle about the trapezium, and on its greatest side describes a segment of a circle similar to those cut off from the circle

1. Trapezia, like this, cut off from an isosceles triangle by a line parallel to the base, occur in the *Papyrus Rhind.*

by the three equal sides.[1] By drawing a diagonal of the trapezium, it will be manifest that the section in question is greater than a semicircle, for the square on this straight line subtending two equal sides of the trapezium must be greater than twice the square on either of them, or than double the square on the third equal side: the square on the greatest side of the trapezium, which is equal to three times the square on any one of the other sides, is therefore

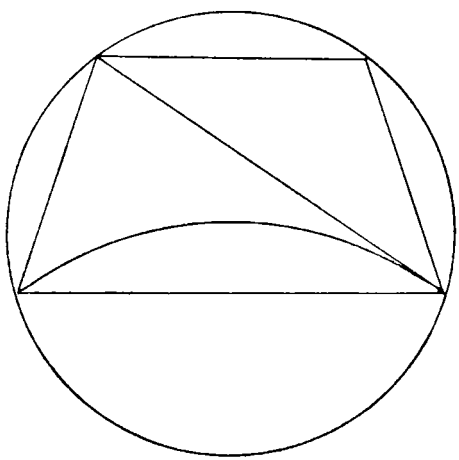


Fig 24.3.

less than the square on the diagonal and the square on the third equal side. Consequently, the angle subtended by the greatest side of the trapezium is acute, and the segment which contains it is, therefore, greater than a semicircle: but this is the exterior boundary of the lune.

'Further, Hippocrates shows that a lune with an exterior arc less than a semicircle can be squared, and gives the following construction for the description of such a lune:[2]

1. Then follows a proof, which I have omitted, that the circle can be circumscribed about the trapezium. This proof is obviously supplied by Simplicius, as is indicated by the change of person . . . as well as by the reference to Euclid, I, 9.

2. Bretschneider, p. 114, notices the archaic manner in which lines and points are denoted in this investigation – . . . and infers from it that Eudemus

'Let $\alpha\beta$ be the diameter of a circle whose centre is κ; let $\gamma\delta$ cut $\beta\kappa$ in the point of bisection γ, and at right angles; through β draw the straight line $\beta\zeta\varepsilon$, so that the part of it, $\zeta\varepsilon$, intercepted between the line $\gamma\delta$ and the circle shall be such that two squares on it shall be equal to three squares on the radius $\beta\kappa$; join $\kappa\zeta$, and produce it to meet the straight line drawn through ε parallel to $\beta\kappa$, and let them meet at η; join $\kappa\zeta$, $\beta\eta$ (these lines will be equal); describe then a circle round the trapezium $\beta\kappa\varepsilon\eta$; also, circum-

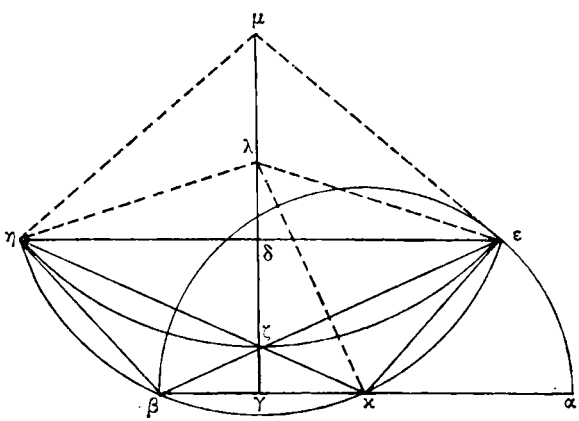


Fig. 24.4.

scribe a circle about the triangle $\varepsilon\zeta\eta$. Let the centres of these circles be λ and μ respectively.

'Now, the segments of the latter circle on $\varepsilon\zeta$ and $\zeta\eta$ are similar to each other, and to each of the segments of the former circle on the equal straight lines $\varepsilon\kappa$, $\kappa\beta$, $\beta\eta$; and, since twice the square on $\varepsilon\zeta$ is equal to three times the square on $\kappa\beta$, the sum of the segments on $\varepsilon\zeta$ and $\zeta\eta$ is equal to the sum of the three segments on $\varepsilon\kappa$, $\kappa\beta$, $\beta\eta$; to each of these equals add the figure bounded by the straight lines $\kappa\varepsilon$, $\kappa\beta$, $\beta\eta$, and the arc $\eta\zeta\varepsilon$, and we shall have the

is quoting the very words of Hippocrates. I have found this observation useful in aiding me to separate the additions of Simplicius from the work of Eudemus.

lune whose exterior arc is $\varepsilon\kappa\beta\eta$ equal to the rectilineal figure composed of the three triangles $\zeta\beta\eta$, $\zeta\beta\kappa$, $\zeta\kappa\varepsilon$.[1]

'That the exterior arc of this lune is smaller than a semicircle, Hippocrates proves, by showing that the angle $\varepsilon\kappa\eta$ lying within the exterior arc of the segment is obtuse, which he does thus: Since the square on $\varepsilon\zeta$ is once and a-half the square on the radius $\beta\kappa$ or $\kappa\varepsilon$, and since, on account of the similarity of triangles $\beta\kappa\varepsilon$ and $\beta\zeta\kappa$, the square on $\kappa\varepsilon$ is greater than twice the square on $\kappa\zeta$, it follows that the square on $\varepsilon\zeta$ is greater than the squares on $\varepsilon\kappa$ and $\kappa\zeta$ together. The angle $\varepsilon\kappa\eta$ is therefore obtuse, and consequently the segment in which it lies is less than a semicircle.'

*

If we examine this oldest fragment of Greek geometry, we see, in the first place, that there is in it a definition of similar segments of circles; they are defined to be those which contain the same quotum of their respective circles, as for instance, a semicircle is similar to a semicircle, the third part of one circle is similar to the third part of another circle.

Next we find the following theorems:

(*a*) Similar segments contain equal angles;

(*b*) These in all semicircles are right; segments which are larger or smaller than semicircles contain, respectively, acute or obtuse angles;

(*c*) The side of a hexagon inscribed in a circle is equal to the radius;

(*d*) In any triangle the square on a side opposite to an acute angle is less than the sum of the squares on the sides which contain the acute angle;

(*e*) In an obtuse-angled triangle the square on the side subtending the obtuse angle is greater than the sum of the squares on the sides containing it;

(*f*) In an isosceles triangle whose vertical angle is double the

1. A pentagon with a re-entrant angle is considered here: but observe – 1st, that it is not called a pentagon, that term being then restricted to the regular pentagon; and, 2nd, that it is described as a rectilineal figure composed of three triangles.

angle of an equilateral triangle, the square on the base is equal to three times the square on one of the equal sides;

(*g*) In equiangular triangles the sides about the equal angles are proportional;

(*h*) Circles are to each other as the squares on their diameters;

(*i*) Similar segments of circles are to each other as the squares on their bases.

Lastly, we observe that the solution of the following problems is required.

(*a*) Construct a square which shall be equal to a given rectilineal figure;

(*b*) Find a line the square on which shall be equal to three times the square on a given line;

(*c*) Find a line such that twice the square on it shall be equal to three times the square on a given line;

(*d*) Being given two straight lines, construct a trapezium such that one of the parallel sides shall be equal to the greater of the two given lines, and each of the three remaining sides equal to the less;

(*e*) About the trapezium so constructed describe a circle;

(*f*) Describe a circle about a given triangle;

(*g*) From the extremity of the diameter of a semicircle draw a chord such that the part of it intercepted between the circle and a straight line drawn at right angles to the diameter at the distance of one-half the radius shall be equal to a given straight line;

(*h*) Describe on a given straight line a segment of a circle which shall be similar to a given one.

25

PLATO

(*c.* 428–*c.* 347 B.C.)

PLATO, one of the greatest and most influential philosophers of all time, was a descendant of the highest ranking families of Athens, tracing his ancestry back on his father's side to Codrus and on his mother's side to the law-giver Solon. He was born at Athens *c.* 428 B.C. An outstanding example of the ancient Greek ideal of universal culture, he applied himself successfully to every branch of science and art, achieving extraordinary skill in gymnastics, as well. Although he was named Aristocles after his grandfather, he was known generally by his nickname, Plato, which is said to have been given to him by his wrestling master because of his strong build and unusually strong shoulders. At the age of twenty Plato attached himself to Socrates with whom he studied for eight years and whose favourite pupil he became. About 400 B.C., after Socrates had been forced to drain the poison cup, Plato, threatened with persecution for his Socratic views, fled from Athens. He continued to travel and study for approximately twelve years, filling out his mathematical and scientific education and taking ever increasing interest in theology and in the theories of the universe and immortality. He remained in Egypt for a long time and there he may have had some contact with the books of Moses.

Invited to Sicily by Dionysius the Elder, Plato, so far from flattering the tyrant of Syracuse, was outspokenly critical. Angry and suspicious, Dionysius allowed himself to be dissuaded from punishing Plato with death, only to sell him into slavery. Ransomed by his friends, Plato returned at length to Athens where he was warmly received. He declined all offers of public office, desiring to devote himself wholly to philosophy. It was at this time that he organized the Academy, a school to which he imparted an impetus so vital and powerful that it was to continue to be an active centre of learning far into the sixth century A.D. Apart from two voyages to Sicily in vain attempts to establish a model government there under Dionysius the Younger, Plato spent his remaining years in Athens writing and teaching.

Convinced that knowledge of mathematics was beneficial in many fields of endeavour and moreover, that it was essential preparation to the understanding of philosophy, Plato required some knowledge of mathematics of any who wished to enter his Academy. Members of the Academy were encouraged by him to undertake mathematical investigations and the advances in mathematics of his century and the next one were all connected directly or indirectly with the Academy. Adored by his scholars, known for his complete dedication to philosophy and for his exemplary moral conduct, Plato inspired the devotion of disciples all over the world. His writings, almost all of them in the form of dialogue, possess in addition to depth of philosophical insight a poetic charm and grace of style which may be said to go far toward explaining the admiration which his works awakened and the excellent state of preservation in which they have come down to us. Plato's *Dialogues* are interspersed everywhere with references to mathematics and with illustrations from mathematics. Taken together, Plato's mathematical illustrations serve to illuminate his philosophy, as was his intention. They present, in addition, a comprehensive view of the problems with which Greek mathematicians were concerned and their mathematical way of thinking.

DIALOGUES OF PLATO

translated by B. Jowett

PHILEBUS

Soc. Then now let us divide the arts of which we were speaking into two kinds; the arts which, like music, are less exact in their results, and those which, like carpentering, are more exact.

Pro. Let us make that division.

Soc. Of the latter class, the most exact of all are those which I mentioned at first.

Pro. I see that you mean arithmetic, and the kindred arts of weighing and measuring.

Soc. Certainly, Protarchus; but are not these also distinguishable into two kinds?

Pro. What are the two kinds?

Soc. In the first place, arithmetic is of two kinds; one of which is popular, and the other philosophical.

Pro. How would you distinguish them?

Soc. There is a wide distinction between them, Protarchus; some arithmeticians reckon unequal units; as for example, two armies, two oxen; the one a very large and the other a very small two. The party who are opposed to them insists that every unit in ten thousand must be the same as every other unit.

Pro. Undoubtedly there is, as you say, a great difference among the votaries of the science; and there may be reasonably supposed to be two sorts of arithmetic.

Soc. And what of the arts of computation and mensuration which are used in building and trading – when we compare them with philosophical geometry and exact calculation, shall we say that they are one or two?

Pro. On the analogy of what has preceded, I should be of opinion that they were two.

Soc. Right; but do you understand why I have discussed the subject?

Pro. I think so, but I should like to be told by you.

Soc. The argument has all along been seeking a parallel to pleasure, and true to that original design, has gone on to ask whether one sort of knowledge is purer than another, as one pleasure is purer than another.

Pro. Clearly; that was the intention.

Soc. And has not the argument in what has preceded, already shown that the arts have different provinces, and vary in their degree of certainty?

Pro. Very true.

Soc. And just now did not the argument first designate a particular art by a common term, thus making us believe in the unity of art; and then again, as if speaking of two different things, proceed to enquire whether the art as pursued by philosophers, or as pursued by non-philosophers, has more of certainty and purity?

Pro. That is the question which the argument is at this moment asking.

Soc. And how, Protarchus, shall we answer the enquirer?

Pro. O, Socrates, we have reached a point at which the difference of clearness in different kinds of knowledge is enormous.

Soc. Then the answer will be the easier.

Pro. Certainly; and let us say in reply, that those arts into which arithmetic and mensuration enter, far surpass all others; and that of these the arts or sciences which are animated by the pure philosophic impulse are infinitely superior in accuracy and truth about measures and numbers.

Soc. Then this is your judgement; and this is the answer which, upon your authority, we will give to all masters of the art of misinterpretation?

Pro. What answer?

Soc. That there are two arts of arithmetic, and two of mensuration; and also several other arts which in like manner have this double nature, and yet only one name.

THEAETETUS

Soc. Then now is the time, my dear Theaetetus, for me to examine, and for you to exhibit; since although Theodorus has praised many a citizen and stranger in my hearing, never did I hear him praise any one as he has been praising you.

Theaet. I am glad to hear it, Socrates; but what if he was only in jest?

Soc. Nay, he is not given to jesting; and I cannot allow you to retract your assent on that ground. For if you do, he will have to clear himself on oath, and I am sure that no one will accuse him of false witness. Do not be shy then, but stand to your word.

Theaet. I will do as you wish.

Soc. In the first place, I should like to ask what you learn of Theodorus: something of geometry, I suppose?

Theaet. Yes.

Soc. And astronomy and harmony and calculation?

Theaet. I do my best.

Soc. Yes, my boy, and so do I; and my desire is to learn of him, and of anybody who seems to understand these things. And I get on pretty well in general; but there is a little matter which I want you and the company to aid me in investigating. Will you answer

me a question: 'Is not learning growing wiser about that which you learn?'

Theaet. Of course.

Soc. And by wisdom the wise are wise?

Theaet. Yes.

Soc. And is that different from knowledge?

Theaet. What is different?

Soc. Wisdom; are not men wise in that which they know?

Theaet. Certainly.

Soc. Then wisdom and knowledge are the same?

Theaet. Yes.

Soc. And this is the very difficulty which I can never explain to myself – What is knowledge? Can we answer that question? What do you say? and which of us will answer first? whoever misses shall sit down, as at a game of ball, and shall be donkey, as the boys say, to the rest of the company; he who lasts out his competitors in the game without missing, shall be our king, and shall have the right of asking any questions which he likes. . . . Why is there no reply? I hope, Theodorus, that I am not betrayed into rudeness by my love of conversation? I only want to make us talk and be friendly and sociable.

Theod. The reverse of rudeness, Socrates: but I would rather that you would ask one of the young fellows; for the truth is, that I am not in the habit of playing at your game of question and answer, and I am too old to learn; the young will be more apt, and they will improve more than I shall, for youth is always able to improve. Having already made a beginning with him, I would advise you to detain Theaetetus, and interrogate him.

Soc. Do you hear, Theaetetus, what Theodorus says? the philosopher, whom you would not like to disobey, and whose word ought to be a command to a young man, bids me interrogate you. Take courage, then, and nobly say what you think that knowledge is.

Theaet. Well, Socrates, I will answer as you and he bid me; and if I make a mistake, you will doubtless correct me.

Soc. We will, if we can.

Theaet. Then, I think that the sciences which I learn from Theodorus – geometry, and those which you just now mentioned

– are knowledge; and I would include the art of the cobbler and other craftsmen; these, all and each of them, are knowledge.

Soc. Too much, Theaetetus, too much; the nobility and liberality of your own nature make you give many and diverse things, when I am asking for one simple thing.

Theaet. What do you mean, Socrates?

Soc. Perhaps nothing. I will endeavour, however, to explain what I believe to be my meaning: When you speak of cobbling, you mean the art of making shoes?

Theaet. That is my meaning.

Soc. And when you speak of carpentering, you mean the art of making wooden implements?

Theaet. Yes.

Soc. In both cases you define the subject-matter of each of the two arts?

Theaet. True.

Soc. But that, Theaetetus, was not the question: we wanted to know not the subjects, nor yet the number of the arts or sciences, for we were not going to count them, but we wanted to know the nature of knowledge in the abstract. Am I not right?

Theaet. Perfectly right.

Soc. Take the following example: Suppose that a person were to ask about some very common and obvious thing – for example, What is clay? and we were to reply, that there is a clay of potters, there is a clay of ovenmakers, there is a clay of brick-makers; would not the answer be ridiculous?

Theaet. Truly.

Soc. In the first place, there would be an absurdity in assuming that he who asked the question would understand from our answer the meaning of the word 'clay', merely because we added 'of the image-makers', or of any other workers. For how can a man understand the name of anything, when he does not know what it is?

Theaet. Of course not.

Soc. Then he who does not know what science or knowledge is, has no knowledge of the art or science of making shoes?

Theaet. None.

Soc. Nor of any other science?

Theaet. No.

Soc. And when a man is asked 'what science or knowledge is', to give as an answer the name of some art or science, is ridiculous; for the question is, 'What is knowledge?' and he replies, 'A knowledge of this and that'.

Theaet. True.

Soc. Moreover, he might answer shortly and simply, but he makes an enormous circuit. For example, when asked about the clay, he might have said simply, that 'clay is moistened earth' – whose clay is not to the point.

Theaet. Yes, Socrates, there is no difficulty as you put the question. You mean, if I am not mistaken, something like what occurred to me and to my friend here, your namesake Socrates, in a recent discussion.

Soc. What was that, Theaetetus?

Theaet. Theodorus was writing out for us something about roots, such as the roots of three or five feet, showing that in linear measurement (i.e. comparing the sides of the squares) they are incommensurable by the unit: he selected the irrational roots of the numbers up to seventeen, but he went no farther; and as there are innumerable roots, the notion occurred to us of attempting to include them all under one name or class.

Soc. And did you find such a class?

Theaet. I think that we did; but I should like to have your opinion.

Soc. Let me hear.

Theaet. We divided all numbers into two classes; those which are made up of equal factors multiplying into one another, which we represented as squares and called squares or equilateral numbers – that was one class.

Soc. Very good.

Theaet. The intermediate numbers, such as three and five, and every other number which is made up of unequal factors, either of a greater multiplied by a less, or of a less multiplied by a greater, and when regarded as a figure, is contained in unequal sides – all these we represented as oblong figures, and called them oblong numbers.

Soc. Capital; and what followed?

Theaet. The lines, or sides, which are the roots of (or whose squares are equal to) the equilateral plane numbers, were called by us lengths or magnitudes; and the lines which are the roots of (or whose squares are equal to) the oblong numbers, were called powers or roots; the reason of this latter name being, that they are commensurable with the others (i.e. with the so-called lengths or magnitudes) not in linear measurement, but in the value of their squares; and the same about solids.

Soc. Excellent, my boy; I think that you fully justify the praises of Theodorus, and that he will not be found guilty of false witness.

Theaet. But I am unable, Socrates, to give you a similar answer about knowledge, which is what you appear to want; and therefore Theodorus is a deceiver after all.

Soc. Well, but suppose that you were running a course, and some one said in praise of you, that he had never known any youth who was as good a runner, and afterwards you were beaten in a race by a grown-up man, who was a great runner – would his praise be any the less true?

Theaet. Certainly not.

Soc. And is the discovery of the nature of knowledge really a little matter, as I just now said, or one requiring great skill?

Theaet. Requiring the greatest, I should say.

Soc. Well, then, be of good cheer; do not say that Theodorus was mistaken about you, but do your best to ascertain the true nature of knowledge, as well as of other things.

Theaet. I am eager enough, Socrates, if that would bring to light the truth.

Soc. Come, you made a good beginning just now; let your own answer about roots be your model, and as you comprehended them all in one class, try and bring the many sorts of knowledge under one definition.

STATESMAN

Str. Then we must suppose that the great and small exist and are discerned in both these ways, and not, as we were saying before only relatively to one another, but there must also be another

comparison of them with the mean or ideal standard; would you like to hear the reason why?

Y. Soc. Certainly.

Str. If we assume the greater to exist only in relation to the less, there will never be any comparison of either with the mean.

Y. Soc. True.

Str. And would not this doctrine be the ruin of all the arts and their creations; would not the art of the Statesman and the aforesaid art of weaving disappear? For all these arts are on the watch against excess and defect, not as unrealities, but as real evils, which occasion a difficulty in action; and the excellence of beauty of every work of art is due to this observance of measure.

Y. Soc. Certainly.

Str. But if the science of the Statesman disappears, there will be no possibility of finding out the royal science.

Y. Soc. Very true.

Str. Well, then, as in the Sophist we extorted the inference that not-being had an existence, because here was the point at which the argument eluded our grasp, so in this we must endeavour to show that the greater and less are not only to be measured with one another, but also have to do with the production of the mean; for if this is not admitted, neither a statesman nor any other man of action can be an undisputed master of his science.

Y. Soc. Yes, we must certainly do again what we did then.

Str. But this, Socrates, is a greater work than the other, of which we only too well remember the length. I think, however, that we may fairly assume something of this sort:

Y. Soc. What?

Str. That we shall some day require this notion of a standard with a view to the demonstration of absolute truth; meanwhile, the argument that the very existence of the arts must be held to depend on the possibility of measuring more or less, not only with one another, but also with a view to the attainment of the mean, seems to afford a grand support and satisfactory proof of the doctrine which we are maintaining; for if there are arts, there is a standard of measure, and if there is a standard of measure, there are arts; but if either is wanting, there is neither.

Y. Soc. True; and what is the next step?

Str. The next step clearly is to divide the art of measurement into two parts, and place in the one part all the arts which measure number, length, depth, breadth, swiftness, with their opposites; and to have another part in which they are measured with the mean, and the fit, and the opportune, and the due, and with all those words, in short, which denote a mean or standard removed from the extremes.

THE REPUBLIC

But what branch of knowledge is there, my dear friend, which is of the desired nature; since all the useful arts were reckoned mean by us?

. . . and yet if music and gymnastic are excluded, and the arts are also excluded, what remains?

Well, I said, there may be nothing left; and then we shall have to take something which is of universal application.

What may that be?

A something which all arts and sciences and intelligences use in common, and which every one ought to learn among the elements of education.

What is that?

The little matter of distinguishing one, two and three – in a word, number and calculation – do not all arts and sciences necessarily partake of them?

Yes.

Then the art of war partakes of them?

To be sure.

Then Palamedes, when he appears in the play, proves Agamemnon ridiculously unfit to be a general. Did you never remark how he declares that he had invented number, and had numbered and set in array the ranks of the army at Troy; which implies that they had never been numbered before, and Agamemnon must be supposed literally to have been incapable of counting his own feet – how could he if he was ignorant of number? And if that is true, what sort of general must he have been?

I should say a very strange one, certainly.

Must not a warrior, then, I said, in addition to his military skill, have a knowledge of arithmetic?

Certainly he must, if he is to have the least understanding of military tactics, or indeed, I should rather say, if he is to be a man at all.

I should like to know whether you have the same notion which I have of this study?

What is your notion?

It appears to me to be a study which leads naturally to reflection, and is of the kind which we are seeking, but has never been rightly used; for it is really of use in drawing us towards being.

Will you explain your meaning? he said.

I will try, I said; and I wish you would consider and help me, and say 'yes' or 'no' when I attempt to distinguish in my own mind what branches of knowledge have this attracting power, in order that we may have clearer proof that arithmetic is one of them.

Explain, he said.

I mean to say that objects of sense are of two kinds; some of them do not invite thought because the sense is an adequate judge of them; while in the case of other objects there is a mistrust of the senses which imperatively demands enquiry.

You must be referring, he said, to the manner in which the senses are imposed upon by distance, and by painting in light and shade.

No, I said, that is not my meaning.

Then what is your meaning?

When speaking of uninviting objects, I mean those which do not pass from one sensation to another; inviting objects are those which give opposite sensations; in this latter case the sense coming upon the object, whether at a distance or near, gives no more vivid idea of anything in particular than of its opposite. An illustration will make my meaning clearer – here are three fingers – a little finger, a second finger and a middle finger.

Very good.

You may suppose that they are seen quite close. And here comes the point.

What is that?

Each of them equally appears a finger, whether seen in the middle or at the extremity, whether white or black, or thick or thin – it makes no difference; a finger is a finger all the same. And in all these cases the ordinary soul is not compelled to ask of thought the question what is a finger? for the sight never intimates to her that a finger is other than a finger.

True.

And therefore, I said, there is nothing here which invites or excites intelligence.

There is not, he said.

But is this equally true of the greatness and smallness of the fingers? Can sight adequately perceive them? and is no difference made by the circumstance that one of the fingers is in the middle and another at the extremity? And in like manner does the touch adequately perceive the qualities of thickness or thinness, or softness or hardness? And so of the other senses; do they give perfect intimations of such matters? Is not their mode of operation rather on this wise – the sense which is concerned with the quality of hardness is necessarily concerned also with the quality of softness, and only intimates to the soul that the same thing is felt to be hard and soft?

Very true, he said.

And must not the soul be perplexed at this intimation of a hard which is also soft? What, again, is the meaning of light and heavy, if that which is light is also heavy, and that which is heavy, light?

Yes, he said, these intimations are very curious and have to be explained.

Yes, I said, and in these perplexities the soul naturally summons to her aid calculation and intelligence, that she may see whether the several objects announced are one or two.

True.

And if they turn out to be two, is not each of them one, and different?

Certainly.

And if each is one, and both are two, she will conceive the two as in a state of division, for if they were undivided they could only be conceived of as one?

True.

The eye certainly did behold both small and great, not divided but confused.

Yes.

Whereas the thinking mind, intending to light up the chaos, was compelled to reverse the process, and look at small and great as separate and not confused.

Very true.

And was not this the beginning of the enquiry 'What is great?' and 'What is small?'

Exactly so.

Here began the distinction of the visible and the intelligible.

Most true.

And that is an illustration of my meaning in describing impressions as inviting to the intellect, or the reverse – the inviting impressions are simultaneous with opposite impressions.

I understand, he said, and agree with you.

And to which class do unity and number belong?

I do not know, he replied.

Think a little and you will see that what has preceded will supply the answer; for if simple unity, and that only, can be adequately perceived by the sight or by any other sense, then, as we were saying in the case of the fingers, there will be nothing to attract towards being; but when there is some contradiction always present, and one is the reverse of one and involves the conception of plurality, then thought begins to be aroused within, and the soul perplexed and wanting to arrive at a decision asks 'What is absolute unity?' And this is the way in which the study of the one has a power of drawing and converting the mind to the contemplation of true being.

And surely, he said, this occurs notably when we look at one, for the same thing is seen by us as one and as infinite in multitude?

Yes, I said; and this being true of one must be equally true of all number?

Certainly.

And all arithmetic and calculation have to do with number?

Yes.

And they are conductors to truth?

Yes, in an eminent degree.

Then this is the sort of knowledge of which we are in search, having a double use, military and philosophical; for the man of war must learn the art of number that he may know how to array his troops, and the philosopher also, because he has to rise out of the sea of change and lay hold of true being, if he would be an arithmetician.

That is true.

And our guardian is both warrior and philosopher?

Certainly.

Then this is a kind of knowledge which legislation may fitly prescribe; and we must endeavour to persuade the principal men of our State to go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers in the mind only; nor again, in the spirit of merchants or traders, with a view to buying or selling, but for the sake of their military use, and of the soul herself; and because this will be the easies way for her to pass from becoming to truth and being.

That is excellent, he said.

Yes, I said, and now having spoken of it, I must add how charming the science is! and in how many ways it conduces to our desired end, if pursued in the spirit of a philosopher, and not of a shopkeeper!

How do you mean?

I mean, as I was saying, that arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number, and rebelling against the introduction of visible or tangible objects into the argument. You know how steadily the masters of the art repel and ridicule any one who attempts to divide absolute unity when he is calculating, and if you divide, they multiply,[1] taking care that one shall continue one and not become lost in fractions.

That is very true.

Now, suppose a person were to say to them: O my friends,

1. Meaning either (1) that they integrate the number because they deny the possibility of fractions; or (2) that division is regarded by them as a process of multiplication, and thus the unity and indivisibility of one is still maintained.

what are these wonderful numbers about which you are reasoning, in which, as you say, there is a unity such as you require, and each unit is equal, invariable, indivisible – what would they answer?

They would answer, as I suppose, that they were speaking of those numbers which are only realized in thought.

Then you see that this knowledge may be truly called necessary, necessitating as it does the use of the pure intelligence in the attainment of pure truth?

Yes; that is a marked characteristic of it.

And have you further remarked, that those who have a natural talent for calculation are generally quick at every other kind of knowledge; and even the dull, if they have had an arithmetical training, gain in quickness, if not in any other way?

Very true, he said.

And indeed, you will not easily find a more difficult study, and not many as difficult.

You will not.

And, for these reasons, arithmetic is a kind of knowledge in which the best natures should be trained, and which must not be given up.

I agree.

Let this then be made one of our subjects of education. And next, shall we enquire whether the kindred science also concerns us?

You mean geometry?

Yes.

Certainly, he said; that part of geometry which relates to war is clearly our concern; for in pitching a camp, or taking up a position, or closing or extending the lines of an army, or any other military manoeuvre, whether in actual battle or on a march, there will be a great difference in a general, accordingly as he is or is not a geometrician.

Yes, I said, but for that purpose a very little of either geometry or calculation will be enough; the question is rather of the higher and greater part of geometry, whether that tends towards the great end – I mean towards the vision of the idea of good; and thither, as I was saying, all things tend which compel the soul to

turn her gaze towards that place, where is the full perfection of being, of which she ought, by all means, to attain the vision.

True, he said.

Then if geometry compels us to view being, it concerns us; if becoming only, it does not concern us?

Yes, that is what we assert.

Nevertheless, such a conception of the science is in flat contradiction to the ordinary language of geometricians, as will hardly be denied by those who have any acquaintance with their study: for they speak of squaring and applying and adding, having in view use only, and absurdly confuse the necessities of geometry with those of daily life; whereas knowledge is the real object of the whole science.

Certainly, he said.

Then must not a further admission be made?

What admission?

The admission that this knowledge at which geometry aims is of the eternal, and not of the perishing and transient.

That, he replied, may be readily allowed, and is true.

Then, my noble friend, geometry will draw the soul towards truth, and create the spirit of philosophy, and raise up that which is now unhappily allowed to fall down.

Nothing will be more effectual.

Then nothing should be more effectually enacted than that the inhabitants of your fair city should learn geometry. Moreover the science has indirect effects, which are not small.

Of what kind are they? he said.

There are the military advantages of which you spoke, I said; and in all departments of study, as experience proves, any one who has studied geometry is infinitely quicker of apprehension than one who has not studied it.

Yes, he said, the difference between a geometrician and one who is not a geometrician is very great indeed.

Then shall we propose this as a second branch of knowledge which our youth will study?

Let us make the proposal, he replied.

And suppose we make astronomy the third – what do you say?

I am strongly inclined to it, he said; the observation of the seasons and of months and years is quite essential to husbandry and navigation, and not less essential to military tactics.

I am amused, I said, at your fear of the world, which makes you guard against the appearance of insisting upon useless studies; and I quite admit the difficulty of believing that in every man there is an eye of the soul which, when by other pursuits lost and dimmed, is by these purified and re-illumined; and is more precious far than ten thousand bodily eyes, for by this alone is truth seen. Now there are two classes of persons: one class who will agree with you and will take your words as a revelation; another class who have no understanding of them, and to whom they will naturally seem to be idle tales. And you had better decide at once with which of the two you are arguing; or perhaps, you will say with neither, and that your chief aim in carrying on the argument is your own improvement; at the same time not grudging to others any benefit which they may derive.

I think that I should prefer to carry on the argument on my own behalf.

Then take a step backward, for we have gone wrong in the order of the sciences.

What was the mistake? he said.

After plane geometry, I said, we took solids in revolution, instead of taking solids in themselves; whereas after the second dimension the third, which is concerned with cubes and dimensions of depth, ought to have followed.

That is true, Socrates; but these subjects seem to be as yet hardly explored.

Why, yes, I said, and for two reasons: in the first place, no government patronizes them, which leads to a want of energy in the study of them, and they are difficult; in the second place, students cannot learn them unless they have a teacher. But then a teacher can hardly be found, and even if he could, as matters now stand, the students, who are very conceited, would not mind him. That, however, would be otherwise if the whole State patronized and honoured these studies; then they would find disciples, and there would be continuous and earnest search, and discoveries would be made; since even now, disregarded as

they are by the world, and maimed of their fair proportions, and although none of their votaries can tell the use of them, still these studies force their way by their natural charm, and very likely they may emerge into light.

Yes, he said, there is a remarkable charm in them. But I do not clearly understand the change in the order. First you began with a geometry of plane surfaces?

Yes, I said.

And you placed astronomy next, and then you made a step backward?

Yes, and I have delayed you by my haste; the ludicrous state of solid geometry made me pass over this branch and go on to astronomy, or motion of solids.

True, he said.

Then assuming that the science now omitted would come into existence if encouraged by the State, let us go on to astronomy, which will be fourth.

The right order, he replied.

GORGIAS

Soc. And yet I do not believe that you really mean to call any of these arts rhetoric; although the precise expression which you used was, that rhetoric is an art which operates and is perfected through the medium of discourse; and an adversary who wished to be captious might take a fancy to say, 'And so, Gorgias, you call arithmetic rhetoric'. But I do not think that you *would* call arithmetic rhetoric, any more than you would call geometry rhetoric.

Gor. You are quite right, Socrates, in your apprehension of my meaning.

Soc. Well, then, let me now have the rest of my answer: – seeing that rhetoric is one of those arts which works mainly by the use of words, and there are other arts which also use words, tell me what is that quality in words with which rhetoric is concerned: – Suppose that a person asks me about some of the arts which I was mentioning just now; he might say, 'Socrates, what is arithmetic?' and I should reply to him, as you replied to me just now, that

arithmetic is one of those arts which is perfected by words. And then he would proceed: 'With what?' and I should say, With the knowledge of odd and even numbers, and how many there are of each. And if he asked again: 'And what is the art of calculation?' I should say, That also is one of the arts which is concerned wholly with words. And if he further said, 'Concerned with what?' I should say in the clerk's phrase, 'as aforesaid', like arithmetic, but with a difference, and the difference is that the art of calculation considers not only the quantities of odd and even numbers, but also their relation to themselves and to one another. And suppose, again, I were to say that astronomy is only words – he would ask, 'Words about what, Socrates?' and I should answer, that astronomy tells us about the motions of the stars and sun and moon, and their relative swiftness.

Gor. Very true, Socrates; I admit that.

Soc. And now let us have from you, Gorgias, the truth about rhetoric: which you would admit (would you not?) to be one of those arts which operate and are perfected through the medium of words?

Gor. True.

Soc. Tell me, I say, what about? To what class of things do the words which rhetoric uses relate?

LAWS

The legislator is to consider all these things, and to bid the citizens, as far as possible, not to lose sight of numerical order; for no single instrument of youthful education has such mighty power, both as regards domestic economy and politics, and in the arts, as the study of arithmetic. Above all, arithmetic stirs up him who is by nature sleepy and dull, and makes him quick to learn, retentive, shrewd, and aided by art divine he makes progress quite beyond his natural powers. All these, if only the legislator, by laws and institutions, can banish meanness and covetousness from the souls of the disciples, and enable them to profit by them, will be excellent and suitable instruments of education. But if he cannot, he will unintentionally create in them, instead of wisdom, the habit of craft, which evil tendency

may be observed in the Egyptians and Phoenicians, and many other races, through the general illiberality of their pursuits and possessions, whether some unworthy legislator of theirs has been the cause, or some impediment of chance or nature.

*

Ath. All freemen, I conceive, should learn as much of these branches of knowledge as every child in Egypt is taught when he learns his alphabet. In that country arithmetical games have been actually invented for the use of children, which they learn as a pleasure and amusement. They have to distribute apples and garlands, using the same number sometimes for a larger and sometimes for a lesser number of persons; and they arrange pugilists and wrestlers as they pair together by lot or remain over, and show the order in which they follow. Another mode of amusing them is by distributing vessels, some in which gold, brass, silver, and the like are mixed, others in which they are unmixed; as I was saying, they adapt to their amusement the numbers in common use, and in this way make more intelligible to their pupils the arrangements and movements of armies and expeditions, and in the management of a household they make people more useful to themselves, and more wide awake; and again in measurements of things which have length, and breadth, and depth, they free us from that natural ignorance of all these things which is so ludicrous and disgraceful.

26

EUCLID OF ALEXANDRIA

(*c.* 300 B.C.)

OUR knowledge of the life of Euclid is meagre and indirect. Even in ancient times there was no account of his life available. It is now strongly believed that he had been a mathematician of note in Athens for some time before his arrival in Alexandria, where, under his direct tutelage and influence, there arose a powerful school of Greek mathematics. In allusions to him in ancient writings, he is rarely mentioned by name, being referred to simply as the author of the *Elements*. Proclus (A.D. 412 85), the great historian on whom we rely for almost all our information concerning the ancient Greek mathematicians, was himself forced to employ inference in arriving at his conclusions that Euclid was younger than the pupils of Plato and older than Archimedes, and therefore that he lived in the time of the first Ptolemy, i.e. *c.* 300 B.C. We learn something of his character from Pappus (*c.* A.D. 300) who, contrasting the attitude of Apollonius and Euclid towards their predecessors, pointedly mentions Euclid's fairness in acknowledging the achievements of others and his kindness to anyone who could in the least degree advance the development of mathematics. Both commentators have been amply supported in their appraisal of Euclid's character and personality by the evidence of Euclid's own writing and collateral works. Euclid is thus known to us not only as the first of the great mathematicians of the Alexandrian school, but also as a kindly teacher, honourable and fair minded, with no trace of acquisitiveness regarding either money or reputation.

Using the method of definitions, axioms, postulates and propositions in logical sequence, Euclid organized the geometrical knowledge extant in his time in thirteen books. The entire treatise was known as the *Elements*. Here he presented a judicious selection of both old and recent works, sometimes rearranging the original order of theorems, and in such cases adding new proofs where the old proofs had become invalid by reason of the rearrangement. In addition to Pythagorean geometry he included the Eudoxian theory of proportion and the method of

exhaustions. He dealt with both commensurable and incommensurable quantities. Although specific contributions original with Euclid are difficult to trace, it is generally agreed that he is the originator of the 'Euclidean algorithm' for finding the greatest common measure of two given numbers, and of the proof of the infinity of primes. Euclid was the first writer on mathematics to include the concept of parallel lines among his postulates.

While the *Elements* is generally conceded to be Euclid's masterpiece other works in mathematics, astronomy, mechanics, music and optics have been attributed to him. Surviving (though in incomplete form) are the *Data* containing problems in geometry apparently intended for those who had completed the study of the *Elements*, the *De Divisionibus*, containing problems in the division of areas, the *Optics*, in two forms, one of which is most probably Euclid's own, and the *Phaenomena*, on the geometry of the sphere.

Through countless editions, translations, commentaries and revisions, Euclid's *Elements* remained for more than two thousand years a most important means whereby students received an introduction to the study of mathematics. During the period of some two hundred years just passed, the methodology displayed by Euclid so insistently and so successfully in his *Elements* has proved itself again and again to be a powerful inspiration in the development of new mathematical systems. However far the subject-matter of a modern system may be removed from the substance of the *Elements*, its structure nevertheless rests on a basis of axiomatic method taught by Euclid.

ELEMENTS

From The Thirteen Books of Euclid's Elements
by Sir Thomas L. Heath

BOOK I

Postulate 5

Καὶ ἐὰν εἰς δύο εὐθεῖας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ' ἄπειρον συμπίπτειν, ἐφ' ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two

straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Although Aristotle gives a clear idea of what he understood by a *postulate*, he does not give any instances from geometry; still less has he any allusion recalling the particular postulates found in Euclid. We naturally infer that the formulation of these postulates was Euclid's own work. There is a more positive indication of the originality of Postulate 5, since in the passage (*Anal. prior*, II, 16, 65 a 4) quoted above in the note on the definition of parallels he alludes to some *petitio principii* involved in the theory of parallels current in his time. This reproach was removed by Euclid when he laid down this epoch-making Postulate. When we consider the countless successive attempts made through more than twenty centuries to prove the Postulate, many of them by geometers of ability, we cannot but admire the genius of the man who concluded that such a hypothesis, which he found necessary to the validity of his whole system of geometry, was really indemonstrable.

From the very beginning, as we know from Proclus, the Postulate was attacked as such, and attempts were made to prove it as a theorem or to get rid of it by adopting some other definition of parallels; while in modern times the literature of the subject is enormous. Riccardi (*Saggio di una bibliografia Euclidea*, Part IV, Bologna, 1890) has twenty quarto pages of titles of monographs relating to Postulate 5 between the dates 1607 and 1887. Max Simon (*Ueber die Entwicklung der Elementar-geometrie im XIX. Jahrhundert*, 1906) notes that he has seen three new attempts, as late as 1891 (a century after Gauss laid the foundation of non-Euclidean geometry), to prove the theory of parallels independently of the Postulate. Max Simon himself (pp. 53–61) gives a large number of references to books or articles on the subject and refers to the copious information, as to contents as well as names, contained in Schotten's *Inhalt und Methode des planimetrischen Unterrichts*, II. 183–332.

*

Alternatives for Postulate 5

It may be convenient to collect here a few of the more

noteworthy substitutes which have from time to time been formally suggested or tacitly assumed.

(1) *Through a given point only one parallel can be drawn to a given straight line* or, *Two straight lines which intersect one another cannot both be parallel to one and the same straight line.*

This is commonly known as 'Playfair's Axiom', but it was of course not a new discovery. It is distinctly stated in Proclus' note to Eucl. 1. 31.

(1*a*) *If a straight line intersect one of two parallels, it will intersect the other also* (Proclus).

(1*b*) *Straight lines parallel to the same straight line are parallel to one another.*

The forms (1*a*) and (1*b*) are exactly equivalent to (1).

(2) *There exist straight lines everywhere equidistant from one another* (Posidonius and Geminus); with which may be compared Proclus' tacit assumption that *Parallels remain, throughout their length, at a finite distance from one another.*

(3) *There exists a triangle in which the sum of the three angles is equal to two right angles* (Legendre).

(4) *Given any figure, there exists a figure similar to it of any size we please* (Wallis, Carnot, Laplace).

Saccheri points out that it is not necessary to assume so much, and that it is enough to postulate that *there exist two unequal triangles with equal angles.*

(5) *Through any point within an angle less than two-thirds of a right angle a straight line can always be drawn which meets both sides of the angle* (Legendre).

With this may be compared the similar axiom of Lorenz (*Grundriss der reinen und angewandten Mathematik*, 1791): *Every straight line through a point within an angle must meet one of the sides of the angle.*

(6) *Given any three points not in a straight line, there exists a circle passing through them* (Legendre, W. Bolyai).

(7) '*If I could prove that a rectilineal triangle is possible the content of which is greater than any given area, I am in a position to prove perfectly rigorously the whole of geometry*' (Gauss, in a letter to W. Bolyai, 1799).

Cf. the proposition of Legendre numbered 3 above, and the

axiom of Worpitzky: *There exists no triangle in which every angle is as small as we please.*

(8) *If in a quadrilateral three angles are right angles, the fourth angle is a right angle also* (Clairaut, 1741).

(9) *If two straight lines are parallel, they are figures opposite to (or the reflex of) one another with respect to the middle points of all their transversal segments* (Veronese, *Elementi*, 1904).

Or, *Two parallel straight lines intercept, on every transversal which passes through the middle point of a segment included between them, another segment the middle point of which is the middle point of the first* (Ingrami, *Elementi*, 1904).

Veronese and Ingrami deduce immediately Playfair's Axiom.

BOOK VII

Proposition 1

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

For, the less of two unequal numbers *AB*, *CD* being continually subtracted from the greater, let the number which is left never measure the one before it until an unit is left;

I say that *AB*, *CD* are prime to one another, that is, that an unit alone measures *AB*, *CD*.

For, if *AB*, *CD* are not prime to one another, some number will measure them.

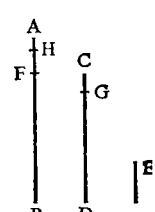


Fig. 26.1.

Let a number measure them, and let it be *E*; let *CD*, measuring *BF*, leave *FA* less than itself, let *AF*, measuring *DG*, leave *GC* less than itself, and let *GC*, measuring *FH*, leave an unit *HA*.

Since then, *E* measures *CD*, and *CD* measures *BF*, therefore *E* also measures *BF*.

But it also measures the whole *BA*; therefore it will also measure the remainder *AF*.

But *AF* measures *DG*; therefore *E* also measures *DG*.

But it also measures the whole DC; therefore it will also measure the remainder CG.

But CG measures FH; therefore E also measures FH.

But it also measures the whole FA; therefore it will also measure the remainder, the unit AH, though it is a number: which is impossible.

Therefore no number will measure the numbers AB, CD; therefore AB, CD are prime to one another. [VII, Def. 12]

Q. E. D.

It is proper to remark here that the representation in Books VII to IX of numbers by straight lines is adopted by Heiberg from the MSS. The method of those editors who substitute *points* for lines is open to objection because it practically necessitates, in many cases, the use of specific numbers, which is contrary to Euclid's manner.

'Let CD, measuring BF, leave FA less than itself.' This is a neat abbreviation for saying, measure along BA successive lengths equal to CD until a point F is reached such that the length FA remaining is less than CD; in other words, let BF be the largest exact multiple of CD contained in BA.

Euclid's method in this proposition is an application to the particular case of prime numbers of the method of finding the greatest common measure of two numbers not prime to one another, which we shall find in the next proposition. With our notation, the method may be shown thus. Supposing the two numbers to be a, b, we have, say,

$$
\begin{array}{l}
b)a(p \\
\underline{pb} \\
\quad c)b(q \\
\quad\ \underline{qc} \\
\quad\quad d)c(r \\
\quad\quad\ \underline{rd} \\
\quad\quad\ \ 1
\end{array}
$$

If now a, b are not prime to one another, they must have a common measure e, where e is some integer, not unity.

And since e measures a, b, it measures $a-pb$, i.e. c.

Again, since e measures b, c, it measures $b-qc$, i.e. d,

and lastly, since e measures c, d, it measures $c-rd$, i.e. 1: which is impossible.

Therefore there is no integer, except unity, that measures a, b, which are accordingly prime to one another.

Observe that Euclid assumes as an axiom that, if a, b are both divisible by c, so is $a-pb$. In the next proposition he assumes as an axiom that c will in the case supposed divide $a+pb$.

Proposition 2

Given two numbers not prime to one another, to find their greatest common measure.

Let AB, CD be the two given numbers not prime to one another.

Thus it is required to find the greatest common measure of AB, CD.

If now CD measures AB – and it also measures itself – CD is a common measure of CD, AB.

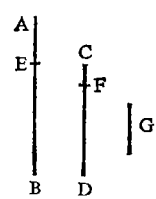


Fig. 26.2.

And it is manifest that it is also the greatest; for no greater number than CD will measure CD.

But, if CD does not measure AB, then, the less of the numbers AB, CD being continually subtracted from the greater, some number will be left which will measure the one before it.

For an unit will not be left; otherwise AB, CD will be prime to one another [VII, 1], which is contrary to the hypothesis.

Therefore some number will be left which will measure the one before it.

Now let CD, measuring BE, leave EA less than itself, let EA, measuring DF, leave FC less than itself, and let CF measure AE.

Since then, CF measures AE, and AE measures DF, therefore CF will also measure DF.

But it also measures itself; therefore it will also measure the whole CD.

But CD measures BE; therefore CF also measures BE.

But it also measures EA; therefore it will also measure the whole BA.

But it also measures CD; therefore CF measures AB, CD.

Therefore CF is a common measure of AB, CD.

I say next that it is also the greatest.

For, if CF is not the greatest common measure of AB, CD, some number which is greater than CF will measure the numbers AB, CD.

Let such a number measure them, and let it be G.

Now, since G measures CD, while CD measures BE, G also measures BE.

But it also measures the whole BA; therefore it will also measure the remainder AE.

But AE measures DF; therefore G will also measure DF.

But it also measures the whole DC; therefore it will also measure the remainder CF, that is, the greater will measure the less: which is impossible.

Therefore no number which is greater than CF will measure the numbers AB, CD; therefore CF is the greatest common measure of AB, CD.

Porism. From this it is manifest that, if a number measure two numbers, it will also measure their greatest common measure.

Q. E. D.

Here we have the exact method of finding the greatest common measure given in the text-books of algebra, including the *reductio ad adsurdum* proof that the number arrived at is not only a common measure but the *greatest* common measure. The process of finding the greatest common measure is simply shown thus:

$$\begin{array}{l} b)\underline{a(p} \\ \quad \underline{pb} \\ \quad c)\underline{b(q} \\ \quad\quad \underline{qc} \\ \quad\quad d)\underline{c(r} \\ \quad\quad\quad \underline{rd} \end{array}$$

We shall arrive, says Euclid, at some number, say d, which measures the one before it, i.e. such that $c=rd$. Otherwise the process would go on until we arrived at unity. This is impossible

because in that case a, b would be prime to one another, which is contrary to the hypothesis.

Next, like the text-books of algebra, he goes on to show that d will be *some* common measure of a, b. For d measures c;
therefore it measures $qc+d$, that is, b,
and hence it measures $pb+c$, that is, a.

Lastly, he proves that d is the *greatest* common measure of a, b as follows.

Suppose that e is a common measure greater than d,

Then e, measuring a. b, must measure $a-pb$, or c.

Similarly e must measure $b-qc$, that is, d: which is impossible, since e is by hypothesis greater than d.

Therefore etc.

Euclid's proposition is thus *identical* with the algebraical proposition generally given, e.g. in Todhunter's algebra, except that of course Euclid's numbers are integers.

Proposition 29

Any prime number is prime to any number which it does not measure.

Let A be a prime number, and let it not measure B; I say that B, A are prime to one another.

For, if B, A are not prime to one another, some number will measure them. ——————A

Let C measure them. ——————————B

Since C measures B, and A does not measure B, therefore C is not the same with A. ——C

Now, since C measures B, A, therefore it also measures A which is prime, though it is not the same with it: which is impossible.

Therefore no number will measure B, A.

Therefore A, B are prime to one another.

Q. E. D.

If a is prime and does not measure b, then a, b are prime to one another. The proof is self-evident.

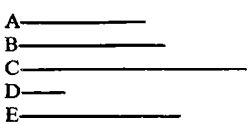
Proposition 30

If two numbers by multiplying one another make some number, and any prime number measure the product, it also measures one of the original numbers.

For let the two numbers A, B by multiplying one another make C, and let any prime number D measure C; I say that D measures one of the numbers A, B.

For let it not measure A.

Now D is prime;



therefore A, D are prime to one another. [VII, 29]

And, as many times as D measures C, so many units let there be in E.

Since then, D measures C according to the units in E, therefore D by multiplying E has made C. [VII, Def. 15]

Further, A by multiplying B has also made C; therefore the product of D, E is equal to the product of A, B.

Therefore, as D is to A, so is B to E. [VII, 19]

But D, A are prime to one another, primes are also least, [VII, 21] and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; therefore D measures B. [VII, 20]

Similarly we can also show that, if D do not measure B, it will measure A.

Therefore D measures one of the numbers A, B.

Q. E. D.

If c, a prime number, measure ab, c will measure either a or b.

Suppose c does not measure a.

Therefore c, a are prime to one another. [VII, 29]

Suppose $ab=mc$.

Therefore $c:a=b:m$. [VII, 19]

Hence [VII, 20, 21] c measures b.

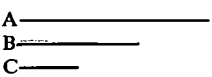
Similarly, if c does not measure b, it measures a.

Therefore it measures one or other of the two numbers a, b.

Proposition 31

Any composite number is measured by some prime number.

Let A be a composite number;



I say that A is measured by some prime number.

For, since A is composite, some number will measure it.

Let a number measure it, and let it be B.

Now, if B is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Let a number measure it, and let it be C.

Then, since C measures B, and B measures A therefore C also measures A.

And, if C is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Thus, if the investigation be continued in this way, some prime number will be found which will measure the number before it, which will also measure A.

For, if it is not found, an infinite series of numbers will measure the number A, each of which is less than the other: which is impossible in numbers.

Therefore some prime number will be found which will measure the one before it, which will also measure A.

Therefore any composite number is measured by some prime number.

BOOK IX

Proposition 20

Prime numbers are more than any assigned multitude of prime numbers.

Let A, B, C be the assigned prime numbers; I say that there are more prime numbers than A, B, C.

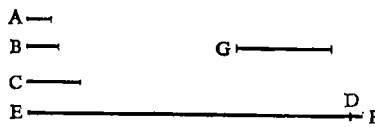
For let the least number measured by *A*, *B*, *C* be taken, and let it be *DE*; let the unit *DF* be added to *DE*.

Then *EF* is either prime or not.

First, let it be prime; then the prime numbers *A*, *B*, *C*, *EF* have been found which are more than *A*, *B*, *C*.

Next, let *EF* not be prime; therefore it is measured by some prime number. [VII, 31]

Let it be measured by the prime number *G*.



I say that *G* is not the same with any of the numbers *A*, *B*, *C*.

For, if possible, let it be so.

Now *A*, *B*, *C* measure *DE*; therefore *G* also will measure *DE*.

But it also measures *EF*.

Therefore *G*, being a number, will measure the remainder, the unit *DF*: which is absurd.

Therefore *G* is not the same with any one of the numbers *A*, *B*, *C*.

And by hypothesis it is prime.

Therefore the prime numbers *A*, *B*, *C*, *G* have been found which are more than the assigned multitude of *A*, *B*, *C*.

Q. E. D.

We have here the important proposition that *the number of prime numbers is infinite.*

The proof will be seen to be the same as that given in our algebraical textbooks. Let *a*, *b*, *c*, . . . *k* be any prime numbers.

Take the product *abc* . . . *k* and add unity.

Then (*abc* . . . *k* + 1) is either a prime number or not a prime number.

(1) If it *is*, we have added another prime number to those given.

(2) If it is *not*, it must be measured by some prime number [VII, 31], say *p*.

Now *p* cannot be identical with any of the prime numbers *a*, *b*, *c*, . . . *k*.

For, if it is, it will divide *abc* . . . *k*.

Therefore, since it divides ($abc \ldots k + 1$) also, it will measure the difference, or unity: which is impossible.

Therefore in any case we have obtained one fresh prime number.

And the process can be carried on to any extent.

27

APOLLONIUS OF PERGA

(Third century B.C.)

APOLLONIUS, an ancient mathematician of the great Alexandrian school, who was known to his contemporaries as 'the Great Geometer', was born at Perga in Pamphylia, a region in the south of Asia Minor. At least twenty-five years younger than Archimedes, he was at the height of his powers during the reign of Ptolemy Euergetes (247–221 B.C.) and Ptolemy Philopater (221–203 B.C.). Although it is believed by some that he appropriated a number of Archimedes' discoveries, he nevertheless has the unquestioned credit for having improved and advanced the theory of conics. Apollonius invented a generalized theory of conic sections in which he replaced the constructions of the conics proposed by Menaechmus (sections made by a plane perpendicular to an element of a right-angled, acute-angled and obtuse-angled cone for the parabola, ellipse and hyperbola, respectively) by constructions exhibiting all the conic sections on a single cone through the variation of the angle of the cutting plane. His detailed and well-organized works swiftly became standard texts in ancient times, superseding previous writings, especially those on conics. Seven of the eight books of his great *Conics* have come down to us, the first four in Greek and the next three in Arabic. Only one treatise (*On the Cutting-off of a Ratio*) of the six Apollonian treatises mentioned by Pappus in his *Treasury of Analysis* has survived. We know from other commentaries that among his lost writings there are works on regular solids, on the theory of irrationals, on a number system, on properties of burning mirrors. Ptolemy credits Apollonius with a theory of epicycles and eccentric circles by which the latter sought to explain the movements of the planets.

Widespread interest in Apollonius's works was aroused by the Latin translation of Books I–IV of the *Conics* by Commanidinus (1566), an edition which also included the lemmas of Pappus and the commentaries of Eutocius. In 1710 a Latin translation of Books I–VII of the *Conics* begun by Gregory and completed by Halley was published. Offering a comprehensive treatment of the extant material of Books

I–VII on a scale hitherto unknown, this publication was further distinguished by the addition of a restoration of Book VIII by Halley who based this part of the work on the accounts of its substance in the commentaries. Many attempts to restore the lost works of Apollonius were made by leading mathematicians, among them Vieta, Snellius, Ghetaldus, Anderson, Schooten, Fermat, Simson, Horsely, Lawson and others. Indeed, the prodigious activity surrounding the study and reconstitution of his works effectively placed Apollonius in the direct line of the invention of analytic geometry and made him a powerful force in the modern development of mathematics.

THE CONIC SECTIONS

From Selections Illustrating the History of Greek Mathematics
with an English Translation by Ivor Thomas

(i.) RELATION TO PREVIOUS WORKS

Eutocius, *Commentary on Apollonius's Conics,*
Apoll. Perg. ed. Heiberg ii. 168. 5–170. 26

Apollonius the geometer, my dear Anthemius, flourished at Perga in Pamphylia during the time of Ptolemy Euergetes, as is related in the life of Archimedes written by Heraclius,[1] who also says that Archimedes first conceived the theorems in conics and that Apollonius, finding they had been discovered by Archimedes but not published, appropriated them for himself, but in my opinion he errs. For in many places Archimedes appears to refer to the elements of conics as an older work, and moreover Apollonius does not claim to be giving his own discoveries; otherwise he would not have described his purpose as 'to investigate these properties more fully and more generally than is done in the works of others'. But what Geminus says is correct: defining a cone as the figure formed by the revolution of a right-angled triangle about one of the sides containing the right angle, the ancients naturally took all cones to be right with one section in each – in the right-angled cone the section now called the parabola, in the obtuse-angled the hyperbola, and in the acute-angled

1. More probably Heraclides.

the ellipse; and in this may be found the reason for the names they gave to the sections. Just as the ancients, investigating each species of triangle separately, proved that there were two right angles first in the equilateral triangle, then in the isosceles, and finally in the scalene, whereas the more recent geometers have proved the general theorem, that *in any triangle the three internal angles are equal to two right angles*, so it has been with the sections of the cone; for the ancients investigated the so-called *section of a right-angled cone* in a right-angled cone only, cutting it by a plane perpendicular to one side of the cone, and they demonstrated the *section of an obtuse-angled cone* in an obtuse-angled cone and the *section of an acute-angled cone* in the acute-angled cone, in the cases of all the cones drawing the planes in the same way perpendicularly to one side of the cone; hence, it is clear, the ancient names of the curves. But later Apollonius of Perga proved generally that all the sections can be obtained in any cone, whether right or scalene, according to different relations of the plane to the cone. In admiration for this, and on account of the remarkable nature of the theorems in conics proved by him, his contemporaries called him the 'Great Geometer'. Geminus relates these details in the sixth book of his *Theory of Mathematics.*

(ii.) SCOPE OF THE WORK

Apollonius, *Conics* i., Preface, Apoll. Perg. ed. Heiberg i. 2. 2–4. 28

Apollonius to Eudemus[1] greeting.

If you are in good health and matters are in other respects as you wish, it is well; I am pretty well too. During the time I spent with you at Pergamum, I noticed how eager you were to make acquaintance with my work in conics; I have therefore sent to you the first book, which I have revised, and I will send the remaining books when I am satisfied with them. I suppose you have not forgotten hearing me say that I took up this study at the request of Naucrates the geometer, at the time when he came to Alexandria and stayed with me, and that, when I had completed

1. Not, of course, the pupil of Aristotle who wrote the famous *History of Geometry*, unhappily lost.

the investigation in eight books, I gave them to him at once, a little too hastily, because he was on the point of sailing, and so I was not able to correct them, but put down everything as it occurred to me, intending to make a revision at the end. Accordingly, as opportunity permits, I now publish on each occasion as much of the work as I have been able to correct. As certain other persons whom I have met have happened to get hold of the first and second books before they were corrected, do not be surprised if you come across them in a different form.

Of the eight books the first four form an elementary introduction. The first includes the methods of producing the three sections and the opposite branches [of the hyperbola] and their fundamental properties, which are investigated more fully and more generally than in the works of others. The second book includes the properties of the diameters and the axes of the sections as well the asymptotes, with other things generally and necessarily used in determining limits of possibility; and what I call diameters and axes you will learn from this book. The third book includes many remarkable theorems useful for the syntheses of solid loci and for determining limits of possibility; most of these theorems, and the most elegant, are new, and it was their discovery which made me realize that Euclid had not worked out the synthesis of the locus with respect to three and four lines, but only a chance portion of it, and that not successfully; for the synthesis could not be completed without the theorems discovered by me. The fourth book investigates how many times the sections of cones can meet one another and the circumference of a circle; in addition it contains other things, none of which have been discussed by previous writers, namely, in how many points a section of a cone or a circumference of a circle can meet [the opposite branches of hyperbolas].

The remaining books are thrown in by way of addition: one of them discusses fully *minima* and *maxima*, another deals with equal and similar sections of cones, another with theorems about the determinations of limits, and the last with determinate conic problems. When they are all published it will be possible for anyone who reads them to form his own judgement. Farewell.

(iii.) DEFINITIONS

Ibid. Definitions, Apoll. Perg. ed. Heiberg i. 6. 2–8. 20

If a straight line be drawn from a point to the circumference of a circle, which is not in the same plane with the point, and be produced in either direction, and if, while the point remains stationary, the straight line be made to move round the circumference of the circle until it returns to the point whence it set out, I call the surface described by the straight line *a conical surface*; it is composed of two surfaces lying vertically opposite to each other, of which each extends to infinity when the straight line which describes them is produced to infinity; I call the fixed point the vertex, and the straight line drawn through this point and the centre of the circle I call the *axis*.

The figure bounded by the circle and the conical surface between the vertex and the circumference of the circle I term a *cone*, and by the *vertex of the cone* I mean the point which is the vertex of the surface, and by the *axis* I mean the straight line drawn from the vertex to the centre of the circle, and by the *base* I mean the circle.

Of cones, I term those *right* which have their axes at right angles to their bases, and *scalene* those which have their axes not at right angles to their bases.

In any plane curve I mean by a *diameter* a straight line drawn from the curve which bisects all straight lines drawn in the curve parallel to a given straight line, and by the *vertex of the curve* I mean the extremity of the straight line on the curve, and I describe each of the parallels as being drawn *ordinatewise* to the diameter.

Similarly, in a pair of plane curves I mean by a *transverse diameter* a straight line which cuts the two curves and bisects all the straight lines drawn in either curve parallel to a given straight line, and by the *vertices of the curves* I mean the extremities of the diameter on the curves; and by an *erect diameter* I mean a straight line which lies between the two curves and bisects the portions cut off between the curves of all straight lines drawn parallel to a given straight line; and I describe each of the parallels as drawn *ordinate-wise to the diameter.*

By *conjugate diameters* in a curve or pair of curves I mean straight lines of which each, being a diameter, bisects parallels to the other.

By an *axis* of a curve or pair of curves I mean a straight line which, being a diameter of the curve or pair of curves, bisects the parallels at right angles.

By *conjugate axes* in a curve or pair of curves I mean straight lines which, being conjugate diameters, bisect at right angles the parallels to each other.

(v.) FUNDAMENTAL PROPERTIES

Ibid., Props. 11–14, Apoll. Perg. ed. Heiberg i. 36. 26–58. 7

Prop. 11

Let a cone be cut by a plane through the axis, and let it be also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and further let the diameter of the section be parallel to one side of the axial triangle; then if any straight line be drawn from the section of the cone parallel to the common section of the cutting plane and the base of the cone as far as the diameter of the section, its square will be equal to the rectangle bounded by the intercept made by it on the diameter in the direction of the vertex of the section and a certain other straight line; this straight line will bear the same ratio to the intercept between the angle of the cone and the vertex of the segment as the square on the base of the axial triangle bears to the rectangle bounded by the remaining two sides of the triangle; and let such a section be called a parabola.

For let there be a cone whose vertex is the point A and whose base is the circle $B\Gamma$, and let it be cut by a plane through the axis, and let the section so made be the triangle $AB\Gamma$, and let it be cut by another plane cutting the base of the cone in the straight line ΔE perpendicular to $B\Gamma$, and let the section so made on the surface of the cone be ΔZE, and let ZH, the diameter of the section, be parallel to $A\Gamma$, one side of the axial triangle, and from the point Z let $Z\Theta$ be drawn perpendicular to ZH, and let $B\Gamma^2 : BA \,.\, A\Gamma = Z\Theta : ZA$, and let any point K be taken at random

on the section, and through K let $K\Lambda$ be drawn parallel to ΔE. I say that $K\Lambda^2 = \Theta Z \cdot E\Lambda$.

For let MN be drawn through Λ parallel to $B\Gamma$; but $K\Lambda$ is parallel to ΔE; therefore the plane through $K\Lambda$, MN is parallel

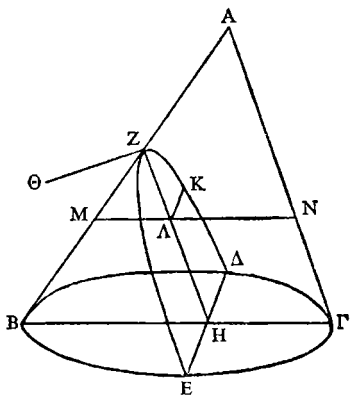


Fig. 27.1.

to the plane through $B\Gamma$, ΔE [Eucl. XI, 15], that is to the base of the cone. Therefore the plane through $K\Lambda$, MN is a circle, whose diameter is MN [Prop. 4]. And $K\Lambda$ is perpendicular to MN, since ΔE is perpendicular to $B\Gamma$ [*Eucl.* XI, 10];

therefore $M\Lambda \cdot \Lambda N = K\Lambda^2$.

And since $B\Gamma^2 : BA \cdot 9\Gamma = \Theta Z : ZA$,

while $B\Gamma^2 : BA \cdot A\Gamma = (B\Gamma : \Gamma A)(B\Gamma : BA)$,

therefore $\Theta Z : ZA = (B\Gamma : \Gamma A)(\Gamma B : BA)$.

But $B\Gamma : \Gamma A = MN : NA$

$= M\Lambda : \Lambda Z$, [*Eucl.* VI, 4

and $B\Gamma : BA = MN : MA$

$= \Lambda M : MZ$ [ibid.

$= N\Lambda : ZA$. [*Eucl.* VI, 2

Therefore $\Theta Z : ZA = (M\Lambda : \Lambda Z)(N\Lambda : ZA)$.

But $(M\Lambda : \Lambda Z)(\Lambda N : ZA) = M\Lambda \cdot \Lambda N : \Lambda Z \cdot ZA$.

Therefore $\Theta Z : ZA = M\Lambda \cdot \Lambda N : \Lambda Z \cdot ZA$.

But $\Theta Z : ZA = \Theta Z \cdot Z\Lambda : \Lambda Z \cdot ZA$,

by taking a common height $Z\Lambda$;

therefore $M\Lambda \cdot \Lambda N : \Lambda Z \cdot ZA = \Theta Z \cdot Z\Lambda : \Lambda Z \cdot ZA$.

Therefore	$M\Lambda . \Lambda N = \Theta Z . Z\Lambda$. [*Eucl.* V, 9
But	$M\Lambda . \Lambda N = K\Lambda^2$;
and therefore	$K\Lambda^2 = \Theta Z . Z\Lambda$.

Let such a section be called a *parabola*, and let ΘZ be called the *parameter of the ordinates* to the diameter ZH, and let it also be called the *erect side* (*latus rectum*).[1]

Prop. 13

Let a cone be cut by a plane through the axis, and let it be cut by another plane meeting each side of the axial triangle, being neither parallel to the base nor subcontrary, and let the plane containing the base of the cone meet the cutting plane in a straight line perpendicular either to the base of the axial triangle or to the base produced; then if a straight line be drawn from any point of the section of the cone parallel to the common section of the planes as far as the diameter of the section, its square will be equal to an area applied to a certain straight line; this line is such that the diameter of the section will bear to it the same ratio as the square on the line drawn from the vertex of the cone parallel to the diameter of the section as far as the base of the triangle bears to the rectangle contained by the intercepts made by it on the sides of the triangle; the breadth of the applied figure will be the intercept made by it on the diameter in the direction of the vertex of the section; and the applied figure will be deficient by a figure similar and similarly situated to the rectangle bounded by the diameter and the parameter; and let such a section be called an ellipse.

Let there be a cone, whose vertex is the point A and whose base is the circle $B\Gamma$, and let it be cut by a plane through the axis, and let the section so made be the triangle $AB\Gamma$, and let it be cut by another plane meeting either side of the axial triangle, being drawn neither parallel to the base nor subcontrary, and let the section made on the surface of the cone be the curve ΔE; let the

1. A *parabola* because the square on the ordinate $K\Lambda$ is *applied* to the parameter ΘZ in the form of the rectangle $\Theta Z \, . \, Z\Lambda$, and is exactly equal to this rectangle. It was Apollonius's most distinctive achievement to have based his treatment of the conic sections on the Pythagorean theory of the *application of areas*.

common section of the cutting plane and of that containing the base of the cone be ZH, perpendicular to $B\Gamma$ and let the diameter of the section be $E\Delta$, and from E let $E\Theta$ be drawn perpendicular to $E\Delta$, and through Λ let AK be drawn parallel to $E\Delta$, and let $AK^2 : BK \cdot K\Gamma = \Delta E : E\Theta$, and let any point Λ be taken on the section, and through Λ let ΛM be drawn parallel to ZH. I say that the square on ΛM is equal to an area applied to the straight line ΘE, having EM for its breadth, and being deficient by a figure similar to the rectangle contained by ΔE, $E\Theta$.

For let $\Delta\Theta$ be joined, and through M let $M\,\Xi\,N$ be drawn parallel to ΘE, and through Θ, Ξ, let ΘN, ΞO be drawn parallel to EM, and through M let ΠMP be drawn parallel to $B\Gamma$. Then

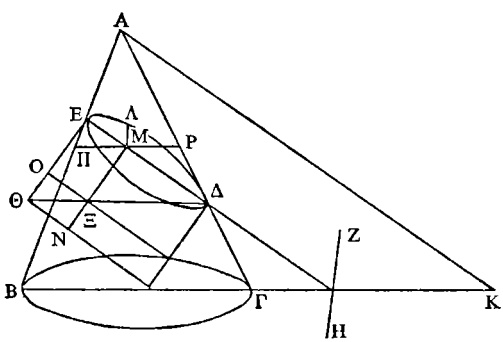


Fig. 27.2.

since ΠP is parallel to $B\Gamma$, and ΛM is parallel to ZH, therefore the plane through ΛM, ΠP is parallel to the plane through ZH, $B\Gamma$ [*Eucl.* xi, 15], that is to the base of the cone. If, therefore, the plane through ΛM, ΠP be produced, the section will be a circle with diameter ΠP [Prop. 4]. And ΛM is perpendicular to it; therefore

$$\Pi M . MP = \Lambda M^2.$$

And since $AK^2 : BK . K\Gamma = E\Delta : E\Theta$,

and $AK^2 : BK . K\Gamma = (AK : KB)(AK : K\Gamma)$,

while $AK : KB = EH : HB$

$= EM : M\Pi$, [*Eucl.* VI, 4

and $AK:K\Gamma = \Delta H:H\Gamma$
$= \Delta M:MP$, [ibid.
therefore $\Delta E:E\Theta = (EM:M\Pi)(\Delta M:MP)$.
But $(EM:M\Pi)(\Delta M:MP) = EM.M\Delta:\Pi M.MP$.
Therefore $EM.M\Delta:\Pi M.MP = \Delta E:E\Theta$
$= \Delta M:M\Xi$. [ibid.
But $\Delta M:M\Xi = \Delta M.ME:\Xi M.ME$,
by taking a common height ME.
Therefore $\Delta M.ME:\Pi M.MP = \Delta M.ME:\Xi M.ME$.
Therefore $\Pi M.MP = \Xi M.ME$. [*Eucl.* V. 9
But $\Pi M.MP = \Lambda M^2$,
as was proved;
and therefore $\Xi M.ME = \Lambda M^2$.

Therefore the square on ΛM is equal to MO, which is applied to ΘE, having EM for its breadth, and being deficient by the figure ON similar to the rectangle ΔE . $E\Theta$. Let such a section be called an *ellipse*, let $E\Theta$ be called *the parameter to the ordinates* to ΔE, and let this line be called the *erect side* (*latus rectum*), and $E\Delta$ the *transverse side*.[1]

Prop. 14

If the vertically opposite surfaces [*of a double cone*] *be cut by a plane not through the vertex, there will be formed on each of the surfaces the section called a hyperbola, and the diameter of both sections will be the same, and the parameter to the ordinates drawn parallel to the straight line in the base of the cone will be equal, and the transverse side of the figure will be common, being the*

1. Let p be the parameter of a conic section and d the corresponding diameter, and let the diameter of the section and the tangent at its extremity be taken as axes of coordinates (in general oblique). Then Props. 11–13 are equivalent to the Cartesian equations.

$$y^2 = px \qquad \text{(the parabola)},$$

and

$$y^2 = px \pm \frac{p}{q}x^2 \text{ (the hyperbola and ellipse respectively).}$$

It is the essence of Apollonius's treatment to express the fundamental properties of the conics as equations between *areas*, whereas Archimedes had given the fundamental properties of the central conics as *proportions*.

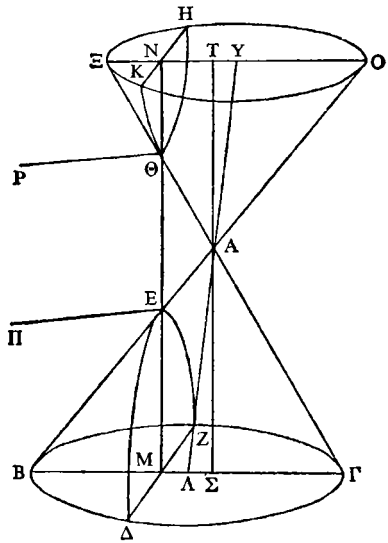


Fig. 27.3.

straight line between the vertices of the sections; and let such sections be called opposite.

Let there be vertically opposite surfaces having the point A for vertex, and let them be cut by a plane not through the vertex, and let the sections so made on the surface be ΔEZ, $H\Theta K$. I say that each of the sections ΔEZ, $H\Theta K$ is the so-called hyperbola.

28

ARCHIMEDES

(*c.* 287–212 B.C.)

ARCHIMEDES, mathematician, physicist and astronomer, the greatest universal intellect of ancient times, who was without a peer in breadth of knowledge and inventiveness for almost two thousand years, was born in Syracuse, a Greek settlement in Sicily, about 287 B.C. The son of Pheidias, an astronomer, he was a kinsman of King Hieron of Syracuse (260–216 B.C.), for whose benefit many of his discoveries in physical and mechanical science were made. The greatest part of Archimedes' life was spent in Syracuse, but it is known that he travelled to Egypt and that he studied for some time in Alexandria where the tradition of Euclid was still strong. Archimedes formed deep and lasting friendships with the mathematicians of Alexandria. His wide acquaintance among the scholars of his time and his personal attachment to some of them is shown in the prefaces written by him to his treatises, *On the Sphere and the Cylinder*, *On Spirals*, *The Method*, and others. Here Archimedes speaks with appreciation of Eratosthenes as 'an earnest student, a man of considerable eminence in philosophy and an admirer of mathematical inquiry', and of Conon of Samos as a mathematician of 'uncommon ability' and 'extraordinary industry' by whose death he suffered the loss of an irreplaceable friend.

In the performance of many services for the king and for Syracuse, Archimedes acquired fame over the whole of the ancient world for his matchless originality in the fields of astronomy, mechanics, hydrostatics and optics as well as in mathematics. His test for determining the proportions of metals in a gold wreath consecrated by the king is well known. He was held in the highest esteem for his cochlias (Archimedean screw), his hydraulic organ, ballistic machines, cranes, burning mirrors, compound pulleys and the like. Nevertheless he regarded such inventions and contrivances as sordidly commercial, refusing to write on these subjects except in one instance (*On Sphere-making*). His published treatises were models of mathematical organization.

When Syracuse was attacked by the Roman forces under Marcellus

Archimedes actively participated in the defence of the city. Directing the use of systems of pulleys, levers, cranes and other devices which he had invented long before, as practical demonstrations of his theories, Archimedes forced the withdrawal of the Romans and was the principal factor in the three-year delay of the fall of Syracuse. Details of Archimedes' death are variously told, but all accounts agree that he was killed at the age of seventy-five by a blundering soldier of the invading Roman army. Marcellus, overwhelmed with grief at the news of his death, attempted to make amends by protecting and even honouring any who could claim relationship to Archimedes. Cicero informs us that when he was quaestor in Sicily (75 B.C.) he saw Archimedes' tombstone which, though neglected, still showed the incised figure of a sphere inscribed in a cylinder. The theory epitomized by this figure was considered by Archimedes to be his greatest achievement.

The discovery by Heiberg in 1906 of a manuscript of Archimedes, treatise, *The Method*, believed until that time to be irretrievably lost, augmented the small portion of his published works which had come down to us. *The Method* is unusually informative for an ancient Greek mathematical work, in that it contains a discussion of the problem of the discovery of mathematical theorems as well as the construction of their proofs. Highly suggestive as Archimedes' works were to his contemporaries and to succeeding generations, it remained for mathematicians living almost two thousand years after him, to perceive the import of the form in which he preferred to state his mathematical problems, and his analyses of them. In the hands of seventeenth-century scholars, Archimedes' writings were of prime influence in the evolution of the calculus.

ON SPIRALS[1]

Archimedes to Dositheus greeting.

Of most of the theorems which I sent to Conon, and of which you ask me from time to time to send you the proofs, the demonstrations are already before you in the books brought to you by Heracleides; and some more are also contained in that which I

1. Extracts from *On Spirals*, *The Method* and *Quadrature of the Parabola* are taken from *The Works of Archimedes*, edited in modern notation by Sir Thomas L. Heath.

now send you. Do not be surprised at my taking a considerable time before publishing these proofs. This has been owing to my desire to communicate them first to persons engaged in mathematical studies and anxious to investigate them. In fact, how many theorems in geometry which have seemed at first impracticable are in time successfully worked out! Now Conon died before he had sufficient time to investigate the theorems referred to; otherwise he would have discovered and made manifest all these things, and would have enriched geometry by many other discoveries besides. For I know well that it was no common ability that he brought to bear on mathematics, and that his industry was extraordinary. But, though many years have elapsed since Conon's death, I do not find that any one of the problems has been stirred by a single person. I wish now to put them in review one by one, particularly as it happens that there are two included among them which are impossible of realization [and which may serve as a warning] how those who claim to discover everything but produce no proofs of the same may be confuted as having actually pretended to discover the impossible.

What are the problems I mean, and what are those of which you have already received the proofs, and those of which the proofs are contained in this book respectively, I think it proper to specify. The first of the problems was, Given a sphere, to find a plane area equal to the surface of the sphere; and this was first made manifest on the publication of the book concerning the sphere, for, when it is once proved that the surface of any sphere is four times the greatest circle in the sphere, it is clear that it is possible to find a plane area equal to the surface of the sphere. The second was, Given a cone or a cylinder, to find a sphere equal to the cone or cylinder; the third, To cut a given sphere by a plane so that the segments of it have to one another an assigned ratio; the fourth, To cut a given sphere by a plane so that the segments of the surface have to one another an assigned ratio; the fifth, To make a given segment of a sphere similar to a given segment of a sphere;[1] the sixth, Given two segments of either the same or

1. *τὸ δοθὲν τμῆμα σφαίρας τῷ δοθέντι τμάματι σφαίρας δμοιώσαι*, i.e. to make a segment of a sphere similar to one given segment and equal in content to another given segment. Cf. *On the Sphere and Cylinder*, II, 5.

different spheres, to find a segment of a sphere which shall be similar to one of the segments and have its surface equal to the surface of the other segment. The seventh was, From a given sphere to cut off a segment by a plane so that the segment bears to the cone which has the same base as the segment and equal height an assigned ratio greater than that of three to two. Of all the propositions just enumerated Heracleides brought you the proofs. The proposition stated next after these was wrong, viz. that, if a sphere be cut by a plane into unequal parts, the greater segment will have to the less the duplicate ratio of that which the greater surface has to the less. That this is wrong is obvious by what I sent you before; for it included this proposition: If a sphere be cut into unequal parts by a plane at right angles to any diameter in the sphere, the greater segment of the surface will have to the less the same ratio as the greater segment of the diameter has to the less, while the greater segment of the sphere has to the less a ratio less than the duplicate ratio of that which the greater surface has to the less, but greater than the sesquialterate[1] of that ratio. The last of the problems was also wrong, viz. that, if the diameter of any sphere be cut so that the square on the greater segment is triple of the square on the lesser segment, and if through the point thus arrived at a plane be drawn at right angles to the diameter and cutting the sphere, the figure in such a form as is the greater segment of the sphere is the greatest of all the segments which have an equal surface. That this is wrong is also clear from the theorems which I before sent you. For it was there proved that the hemisphere is the greatest of all the segments of a sphere bounded by an equal surface.

After these theorems the following were propounded concerning the cone.[2] If a section of a right-angled cone [a parabola], in which the diameter [axis] remains fixed, be made to revolve so that the diameter [axis] is the axis [of revolution], let the figure described by the section of the right-angled cone be called a *conoid*. And if a plane touch the conoidal figure and another plane drawn parallel to the tangent plane cut off a segment of the

1. *(λόγον) μείζονα ἢ ἡμιόλιον τοῦ, ὃν ἔχει κ.τ.λ.*, i.e. a ratio greater than (the ratio of the surfaces).3/2 See *On the Sphere and Cylinder*, II, 8.

2. This should be presumably 'the *conoid*', not 'the cone'.

conoid, let the *base* of the segment cut off be defined as the cutting plane, and the *vertex* as the point in which the other plane touches the conoid. Now, if the said figure be cut by a plane at right angles to the axis, it is clear that the section will be a circle; but it needs to be proved that the segment cut off will be half as large again as the cone which has the same base as the segment and equal height. And if two segments be cut off from the conoid by planes drawn in any manner, it is clear that the sections will be sections of acute-angled cones [ellipses] if the cutting planes be not at right angles to the axis; but it needs to be proved that the segments will bear to one another the ratio of the squares on the lines drawn from their vertices parallel to the axis to meet the cutting planes. The proofs of these propositions are not yet sent to you.

After these came the following propositions about the *spiral*, which are as it were another sort of problem having nothing in common with the foregoing; and I have written out the proofs of them for you in this book. They are as follows. If a straight line of which one extremity remains fixed be made to revolve at a uniform rate in a plane until it returns to the position from which it started, and if, at the same time as the straight line revolves, a point move at a uniform rate along the straight line, starting from the fixed extremity, the point will describe a spiral in the plane. I say then that the area bounded by the spiral and the straight line which has returned to the position from which it started is a third part of the circle described with the fixed point as centre and with radius the length traversed by the point along the straight line during the one revolution. And, if a straight line touch the spiral at the extreme end of the spiral, and another straight line be drawn at right angles to the line which has revolved and resumed its position from the fixed extremity of it, so as to meet the tangent, I say that the straight line so drawn to meet it is equal to the circumference of the circle. Again, if the revolving line and the point moving along it make several revolutions and return to the position from which the straight line started, I say that the area added by the spiral in the third revolution will be double of that added in the second, that in the fourth three times, that in the fifth four times, and generally the

areas added in the later revolutions will be multiples of that added in the second revolution according to the successive numbers, while the area bounded by the spiral in the first revolution is a sixth part of that added in the second revolution. Also, if on the spiral described in one revolution two points be taken and straight lines be drawn joining them to the fixed extremity of the revolving line, and if two circles be drawn with the fixed point as centre and radii the lines drawn to the fixed extremity of the straight line, and the shorter of the two lines be produced, I say that (1) the area bounded by the circumference of the greater circle in the direction of (the part of) the spiral included between the straight lines, the spiral (itself) and the produced straight line will bear to (2) the area bounded by the circumference of the lesser circle, the same (part of the) spiral and the straight line joining their extremities the ratio which (3) the radius of the lesser circle together with two-thirds of the excess of the radius of the greater circle over the radius of the lesser bears to (4) the radius of the lesser circle together with one-third of the said excess.

The proofs then of these theorems and others relating to the spiral are given in the present book. Prefixed to them, after the manner usual in other geometrical works, are the propositions necessary to the proofs of them. And here, too, as in the books previously published, I assume the following lemma, that, if there be (two) unequal lines or (two) unequal areas, the excess by which the greater exceeds the less can, by being [continually] added to itself, be made to exceed any given magnitude among those which are comparable with [it and with] one another.

PROPOSITION 1

If a point move at a uniform rate along any line, and two lengths be taken on it, they will be proportional to the times of describing them.

PROPOSITION 2

If each of two points on different lines respectively move along them each at a uniform rate, and if lengths be taken, one on each line, forming pairs, such that each pair are described in equal times, the lengths will be proportionals.

PROPOSITION 3

Given any number of circles, it is possible to find a straight line greater than the sum of all their circumferences.

For we have only to describe polygons about each and then take a straight line equal to the sum of the perimeters of the polygons.

PROPOSITION 4

Given two unequal lines, viz. a straight line and the circumference of a circle, it is possible to find a straight line less than the greater of the two lines and greater than the less.

For, by the lemma, the excess can, by being added a sufficient number of times to itself, be made to exceed the lesser line.

Thus e.g. if $c>l$ (where c is the circumference of the circle and l the length of the straight line), we can find a number n such that

$$n(c-l)>l.$$

Therefore
$$c-l>\frac{l}{n},$$

and
$$c>l+\frac{l}{n}>l.$$

Hence we have only to divide l into n equal parts and add one of them to l. The resulting line will satisfy the condition.

PROPOSITION 5

Given a circle with centre O, and the tangent to it at a point A, it is possible to draw from O a straight line OPF, meeting the circle in P and the tangent in F, such that, if c be the circumference of any given circle whatever,

$$FP : OP < (\text{arc } AP) : c.$$

Take a straight line, as D, greater than the circumference c. [Proposition 3]

Through O draw OH parallel to the given tangent, and draw through A a line APH, meeting the circle in P and OH in H, such that the portion PH intercepted between the circle and the line

OH may be equal to D. Join OP and produce it to meet the tangent in F.

Then $FP : OP = AP : PH$, by parallels,
$= AP : D$
$<$(arc AP) : c.

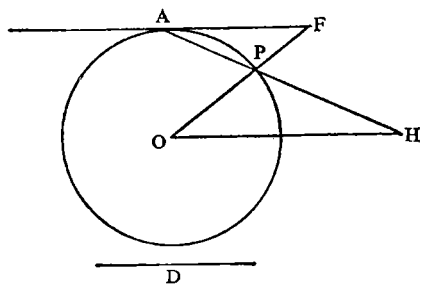


Fig. 28.1.

PROPOSITION 10

If A_1, A_2, A_3 . . . A_n be n lines forming an ascending arithmetical progression in which the common difference is equal to A_1, the least term, then

$$(n+1)A_n^2 + A_1(A_1 + A_3 + \ldots + A_n) = 3(A_1^2 + A_2^2 + \ldots + A_n^2).$$

[Archimedes' result is equivalent to

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.]$$

Cor. 1. *It follows from this proposition that*

$$n \cdot A_n^2 < 3(A_1^2 + A_2^2 + \ldots + A_n^2),$$

and also that

$$n \cdot A_n^2 > 3(A_1^2 + A_2^2 + \ldots + A_{n-1}^2).$$

Cor. 2. *All the results will equally hold if similar figures are substituted for squares.*

Definitions

1. If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line

revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a *spiral* (ἕλιξ) in the plane.

2. Let the extremity of the straight line which remains fixed while the straight line revolves be called the *origin*[1] *(ἀρχά)* of the spiral.

3. And let the position of line from which the straight line began to revolve be called the *initial line*[1] in the revolution (ἀρχὰ τᾶς περιφορᾶς).

4. Let the length which the point that moves along the straight line describes in one revolution be called the *first distance*, that which the same point describes in the second revolution the *second distance*, and similarly let the distances described in further revolutions be called after the number of the particular revolution.

5. Let the area bounded by the spiral described in the first revolution and the *first distance* be called the *first area*, that bounded by the spiral described in the second revolution and the *second distance* the *second area*, and similarly for the rest in order.

6. If from the origin of the spiral any straight line be drawn, let that side of it which is in the same direction as that of the revolution be called *forward* (προαγόμενα), and that which is in the other direction *backward* (ἑπόμενα).

7. Let the circle drawn with the *origin* as centre and the *first distance* as radius be called the *first circle*, that drawn with the same centre and twice the radius the *second circle*, and similarly for the succeeding circles.

PROPOSITION 12

If any number of straight lines drawn from the origin to meet the spiral make equal angles with one another, the lines will be in arithmetical progression.

1. The literal translation would of course be the 'beginning of the spiral' and 'the beginning of the revolution' respectively. But the modern names will be more suitable for use later on, and are therefore employed here.

PROPOSITIONS 21, 22, 23

Given an area bounded by any arc of a spiral and the lines joining the extremities of the arc to the origin, it is possible to circumscribe about the area one figure, and to inscribe in it another figure, each consisting of similar sectors of circles, and such that the circumscribed figure exceeds the inscribed by less than any assigned area.

For let *BC* be any arc of the spiral, *O* the origin. Draw the circle with centre *O* and radius *OC*, where *C* is the 'forward' end of the arc.

Then, by bisecting the angle *BOC*, bisecting the resulting

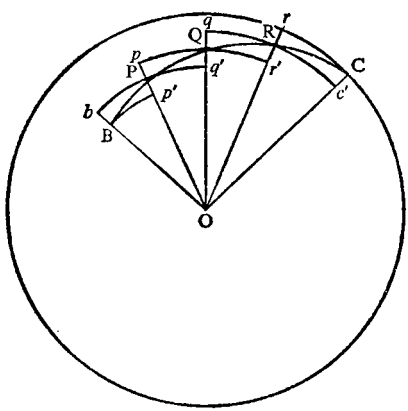


Fig. 28.2.

angles, and so on continually, we shall ultimately arrive at an angle *COr* cutting off a sector of the circle less than any assigned area. Let *COr* be this sector.

Let the other lines dividing the angle *BOC* into equal parts meet the spiral in *P*, *Q*, and let *Or* meet it in *R*. With *O* as centre and radii *OB*, *OP*, *OQ*, *OR* respectively describe arcs of circles Bp', bBq', pQr', qRc', each meeting the adjacent radii as shown in the figure. In each case the arc in the 'forward' direction from each point will fall within, and the arc in the 'backward' direction outside, the spiral.

We have now a circumscribed figure and an inscribed figure

each consisting of similar sectors of circles. To compare their areas, we take the successive sectors of each, beginning from *OC*, and compare them.

The sector *OCr* in the circumscribed figure stands alone.

And (sector ORq)=(sector ORc'),
(sector OQp)=(sector OQr'),
(sector OPb)=(sector OPq'),

while the sector OBp' in the inscribed figure stands alone.

Hence, if the equal sectors be taken away, the difference between the circumscribed and inscribed figures is equal to the difference between the sectors *OCr* and OBp'; and this difference is less than the sector *OCr*, which is itself less than any assigned area.

The proof is exactly the same whatever be the number of angles into which the angle *BOC* is divided, the only difference being that, when the arc begins from the origin, the smallest sectors *OPb*, OPq' in each figure are equal, and there is therefore no inscribed sector standing by itself, so that the difference between the circumscribed and inscribed figures is equal to the sector *OCr* itself.

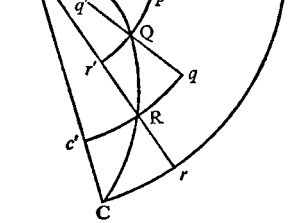


Fig. 28.3.

Thus the proposition is universally true.

Cor. Since the area bounded by the spiral is intermediate in magnitude between the circumscribed and inscribed figures, it follows that

(1) *a figure can be circumscribed to the area such that it exceeds the area by less than any assigned space,*

(2) *a figure can be inscribed such that the area exceeds it by less than any assigned space.*

PROPOSITION 24

The area bounded by the first turn of the spiral and the initial line is equal to one-third of the 'first circle' [$=\frac{1}{3}\pi(2\pi a)^2$, where the spiral is $r=a\theta$].

THE METHOD OF ARCHIMEDES TREATING OF MECHANICAL PROBLEMS

Archimedes to Eratosthenes greeting.

I sent you on a former occasion some of the theorems discovered by me, merely writing out the enunciations and inviting you to discover the proofs, which at the moment I did not give. The enunciations of the theorems which I sent were as follows.

1. If in a right prism with parallelogrammic base a cylinder be inscribed which has its bases in the opposite parallelograms,[1] and its sides [i.e. four generators] on the remaining planes (faces) of the prism, and if through the centre of the circle which is the base of the cylinder and (through) one side of the square in the plane opposite to it a plane be drawn, the plane so drawn will cut off from the cylinder a segment which is bounded by two planes and the surface of the cylinder, one of the two planes being the plane which has been drawn and the other the plane in which the base of the cylinder is, and the surface being that which is between the said planes; and the segment cut off from the cylinder is one-sixth part of the whole prism.

2. If in a cube a cylinder be inscribed which has its bases in the opposite parallelograms[2] and touches with its surface the remaining four planes (faces), and if there also be inscribed in the same cube another cylinder which has its bases in other parallelograms and touches with its surface the remaining four planes (faces), then the figure bounded by the surfaces of the cylinders, which is within both cylinders, is two-thirds of the whole cube.

1. The parallelograms are apparently *squares*.
2. i.e. squares.

Now these theorems differ in character from those communicated before; for we compared the figures then in question, conoids and spheroids and segments of them, in respect of size, with figures of cones and cylinders: but none of those figures have yet been found to be equal to a solid figure bounded by planes; whereas each of the present figures bounded by two planes and surfaces of cylinders is found to be equal to one of the solid figures which are bounded by planes. The proofs then of these theorems I have written in this book and now send to you. Seeing moreover in you, as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer [of mathematical inquiry], I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely that the cone is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure though he did not prove it. I am myself in the position of having first made the discovery of the theorem now to be published [by the method indicated], and I deem it necessary to expound the method partly because I have already spoken of it[1] and I do not want to be thought to have uttered vain words, but equally because I am persuaded that it will be of no little service to mathematics; for I apprehend that some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover

1. Cf. Preface to *Quadrature of Parabola.*

other theorems in addition, which have not yet occurred to me.

First then I will set out the very first theorem which became known to me by means of mechanics, namely that

Any segment of a section of a right-angled cone (i.e. a parabola) is four-thirds of the triangle which has the same base and equal height,

and after this I will give each of the other theorems investigated by the same method. Then, at the end of the book, I will give the geometrical [proofs of the propositions] . . .

[I premise the following propositions which I shall use in the course of the work.]

1. If from [one magnitude another magnitude be subtracted which has not the same centre of gravity, the centre of gravity of the remainder is found by] producing [the straight line joining the centres of gravity of the whole magnitude and of the subtracted part in the direction of the centre of gravity of the whole] and cutting off from it a length which has to the distance between the said centres of gravity the ratio which the weight of the subtracted magnitude has to the weight of the remainder. [*On the Equilibrium of Planes*, I, 8]

2. If the centres of gravity of any number of magnitudes whatever be on the same straight line, the centre of gravity of the magnitude made up of all of them will be on the same straight line. [Cf. Ibid. I, 5]

3. The centre of gravity of any straight line is the point of bisection of the straight line. [Cf. Ibid. I, 4]

4. The centre of gravity of any triangle is the point in which the straight lines drawn from the angular points of the triangle to the middle points of the (opposite) sides cut one another. [Ibid. I, 13, 14]

5. The centre of gravity of any parallelogram is the point in which the diagonals meet. [Ibid. I, 10]

6. The centre of gravity of a circle is the point which is also the centre [of the circle].

7. The centre of gravity of any cylinder is the point of bisection of the axis.

8. The centre of gravity of any cone is [the point which divides

its axis so that] the portion [adjacent to the vertex is] triple [of the portion adjacent to the base].

[All these propositions have already been] proved.[1] [Besides these I require also the following proposition, which is easily proved:

If in two series of magnitudes those of the first series are, in order, proportional to those of the second series and further] the magnitudes [of the first series], either all or some of them, are in any ratio whatever [to those of a third series], and if the magnitudes of the second series are in the same ratio to the corresponding magnitudes [of a fourth series], then the sum of the magnitudes of the first series has to the sum of the selected magnitudes of the third series the same ratio which the sum of the magnitudes of the second series has to the sum of the (correspondingly) selected magnitudes of the fourth series. [*On Conoids and Spheroids*, Prop. 1.]

PROPOSITION 1

Let ABC be a segment of a parabola bounded by the straight line AC and the parabola ABC, and let D be the middle point of AC. Draw the straight line DBE parallel to the axis of the parabola and join AB, BC.

Then shall the segment ABC be 4/3 of the triangle ABC.

From A draw AKF parallel to DE, and let the tangent to the parabola at C meet DBE in E and AKF in F. Produce CB to meet AF in K, and again produce CK to H, making KH equal to CK.

Consider CH as the bar of a balance, K being its middle point.

Let MO be any straight line parallel to ED, and let it meet CF, CK, AC in M, N, O and the curve in P.

Now since CE is a tangent to the parabola and CD the semi-ordinate,

$$EB=BD;$$

'for this is proved in the Elements [of Conics]'.[2]

1. The problem of finding the centre of gravity of a cone is not solved in any extant work of Archimedes. It may have been solved either in a separate treatise [*On Balances* or *Levers*] which is lost, or perhaps in a larger mechanical work of which the extant books *On the Equilibrium of Planes* formed only a part. 2. i.e. the works on conics by Aristaeus and Euclid.

Since FA, MO are parallel to ED, it follows that

$$FK=KA,\ MN=NO.$$

Now, by the property of the parabola, 'proved in a lemma,'

$$MO:OP=CA:AO \text{ [Cf. } \textit{Quadrature of Parabola}\text{, Prop. 5]}$$
$$=CK:KN \text{ [Eucl. VI, 2]}$$
$$=HK:KN.$$

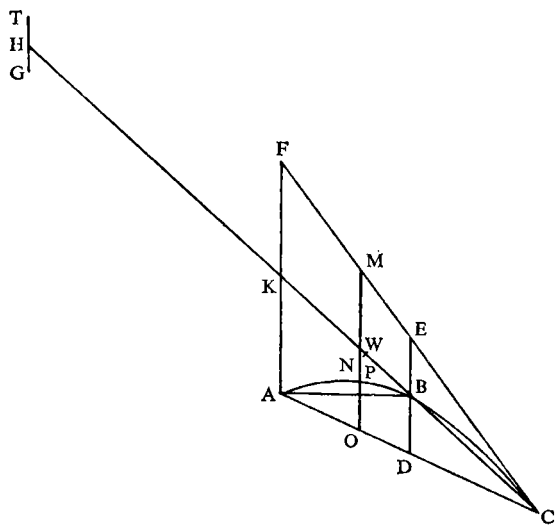


Fig. 28.4.

Take a straight line TG equal to OP, and place it with its centre of gravity at H, so that $TH=HG$; then, since N is the centre of gravity of the straight line MO,

and $$MO:TG=HK:KN,$$

it follows that TG at H and MO at N will be in equilibrium about K. [*On the Equilibrium of Planes*, I, 6, 7]

Similarly, for all other straight lines parallel to DE and meeting the arc of the parabola, (1) the portion intercepted between FC, AC with its middle point on KC and (2) a length equal to the intercept between the curve and AC placed with centre of gravity at H will be in equilibrium about K.

Therefore K is the centre of gravity of the whole system consisting (1) of all the straight lines as MO intercepted between FC, AC and placed as they actually are in the figure and (2) of all the straight lines placed at H equal to the straight lines as PO intercepted between the curve and AC.

And, since the triangle CFA is made up of all the parallel lines like MO, and the segment CBA is made up of all the straight lines like PO within the curve, it follows that the triangle, placed where it is in the figure, is in equilibrium about K with the segment CBA placed with its centre of gravity at H.

Divide KC at W so that $CK=3KW$; then W is the centre of gravity of the triangle ACF; 'for this is proved in the books on equilibrium'
(ἐν τοῖς ἰσορροπικοῖς) [Cf. *On the Equilibrium of Planes* I. 15]

Therefore ΔACF: (segment ABC) $= HK : KW$
$= 3:1.$

Therefore segment $ABC = \frac{1}{3}\Delta ACF.$

But $\Delta ACF = 4\Delta ABC.$

Therefore segment $ABC = \frac{4}{3}\Delta ABC.$

Now the fact here stated is not actually demonstrated by the argument used; but that argument has given a sort of indication that the conclusion is true. Seeing then that the theorem is not demonstrated, but at the same time suspecting that the conclusion is true, we shall have recourse to the geometrical demonstration which I myself discovered and have already published.

PROPOSITION 2

We can investigate by the same method the propositions that

(1) *Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius; and*

(2) *the cylinder with base equal to a great circle of the sphere and height equal to the diameter is* $1\frac{1}{2}$ *times the sphere.*

QUADRATURE OF THE PARABOLA

Archimedes to Dositheus greeting.

When I heard that Conon, who was my friend in his lifetime, was dead, but that you were acquainted with Conon and withal versed in geometry, while I grieved for the loss not only of a friend but of an admirable mathematician, I set myself the task of communicating to you, as I had intended to send to Conon, a certain geometrical theorem which had not been investigated before but has now been investigated by me, and which I first discovered by means of mechanics and then exhibited by means of geometry. Now some of the earlier geometers tried to prove it possible to find a rectilineal area equal to a given circle and a given segment of a circle; and after that they endeavoured to square the area bounded by the section of the whole cone and a straight line, assuming lemmas not easily conceded, so that it was recognized by most people that the problem was not solved. But I am not aware that any one of my predecessors has attempted to square the segment bounded by a straight line and a section of a right-angled cone [a parabola], of which problem I have now discovered the solution. For it is here shown that every segment bounded by a straight line and a section of a right-angled cone [a parabola] is four-thirds of the triangle which has the same base and equal height with the segment, and for the demonstration of this property the following lemma is assumed: that the excess by which the greater of [two] unequal areas exceeds the less can, by being added to itself, be made to exceed any given finite area. The earlier geometers have also used this lemma; for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and further that every pyramid is one-third part of the prism which has the same base with the pyramid and equal height; also, that every cone is one-third part of the cylinder having the same base as the cone and equal height they proved by assuming a certain lemma similar to that aforesaid. And, in the result, each of the aforesaid theorems has been accepted no less than those proved without the lemma. As therefore my work now published

has satisfied the same test as the propositions referred to, I have written out the proof and send it to you, first as investigated by means of mechanics, and afterwards too as demonstrated by geometry. Prefixed are, also, the elementary propositions in conics which are of service in the proof (*στοιχεῖα κωνικὰ χρείαν ἔχοντα ἐς τὰν ἀπόδειξιν*). Farewell.

*

Def. In segments bounded by a straight line and any curve I call the straight line the *base*, and the *height* the greatest perpendicular drawn from the curve to the base of the segment, and the *vertex* the point from which the greatest perpendicular is drawn.

PROPOSITION 20

If Qq be the base, and P the vertex, of a parabolic segment, then the triangle PQq is greater than half the segment PQq.

For the chord Qq is parallel to the tangent at P, and the triangle PQq is half the parallelogram formed by Qq, the tangent at P, and the diameters through Q, q.

Therefore the triangle PQq is greater than half the segment.

Cor. It follows that *it is possible to inscribe in the segment a polygon such that the segments left over are together less than any assigned area.*

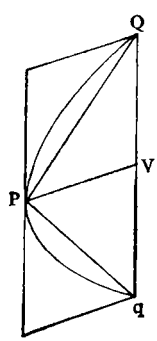


Fig. 28.5.

PROPOSITION 23

Given a series of areas A, B, C, D, . . . Z, of which A is the greatest, and each is equal to four times the next in order, then

$$A+B+C+\ldots+Z+\tfrac{1}{3}Z=\tfrac{4}{3}A$$

Take areas b, c, d, . . . such that

$$b=\tfrac{1}{3}B,$$
$$c=\tfrac{1}{3}C,$$
$$d=\tfrac{1}{3}D, \text{ and so on.}$$

Then, since $b=\frac{1}{3}B$

and $B=\frac{1}{4}A$

$B+b=\frac{1}{3}A$

Similarly $C+c=\frac{1}{3}B$

. . . . . .

Therefore

$$B+C+D+\ldots+Z+b+c+d+\ldots+z$$
$$=\tfrac{1}{3}(A+B+C+\ldots+Y)$$

But $b+c+d+\ldots+y=\frac{1}{3}(B+C+D+\ldots+Y)$

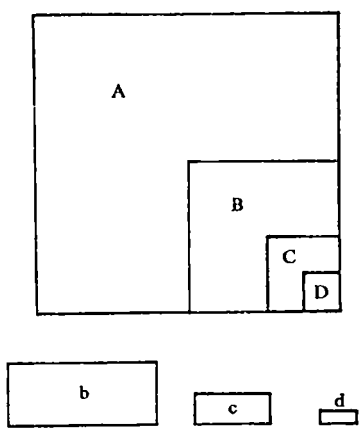


Fig. 28.6.

Therefore, by subtraction,

$$B+C+D+\ldots+Z+z=\tfrac{1}{3}A$$
$$A+B+C+\ldots+Z+\tfrac{1}{3}Z=\tfrac{4}{3}A$$

[The algebraical equivalent of this result is of course

$$1+\tfrac{1}{4}+(\tfrac{1}{4})^2+\ldots+(\tfrac{1}{4})^{n-1}=\tfrac{4}{3}-\tfrac{1}{3}(\tfrac{1}{4})^{n-1}=\frac{1-(\tfrac{1}{4})^n}{1-\tfrac{1}{4}}.\]$$

PROPOSITION 24

Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height.

Suppose $K=\frac{4}{3}\Delta PQq$

where P is the vertex of the segment; and we have then to prove that the area of the segment is equal to K.

For, if the segment be not equal to K, it must either be greater or less.

I. Suppose the area of the segment greater than K.

If then we inscribe in the segments cut off by PQ, Pq triangles which have the same base and equal height, i.e. triangles with the same vertices R, r as those of the segments, and if in the remaining segments we inscribe triangles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment PQq exceeds K.

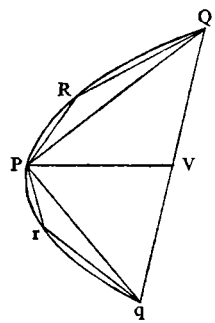


Fig. 28.7.

Therefore the polygon so formed must be greater than the area K; which is impossible, since [Prop. 23]

$$A+B+C+\ \ldots\ +Z<\tfrac{4}{3}A,$$

where $A=\Delta PQq$.

Thus the area of the segment cannot be greater than K.

II. Suppose, if possible, that the area of the segment is less than K.

If then $\Delta PQq=A$, $B=\frac{1}{4}A$, $C=\frac{1}{4}B$, and so on, until we arrive at an area X such that X is less than the difference between K and the segment, we have

$$A+B+C+\ \ldots\ +X+\tfrac{1}{3}X=\tfrac{4}{3}A=K.\ \text{[Prop. 23]}$$

Now, since K exceeds $A+B+C+\ \ldots\ +X$ by an area less than X, and the area of the segment by an area greater than X, it follows that

$$A+B+C+\ \ldots\ +X>(\text{the segment});$$

which is impossible, by Prop. 22 above.

Hence the segment is not less than K.

Thus, since the segment is neither greater nor less than K,

$$(\text{area of segment } PQq)=K=\tfrac{4}{3}\Delta PQq.$$

29

PAPPUS

(*c.* A.D. 300)

PAPPUS, a Greek geometer and scientist who lived in the reign of Diocletian (Roman Emperor, 284–305), was a prolific writer on mathematical and scientific subjects, and a great teacher of the second Alexandrian school. Through the large numbers of pupils attracted to his side, he was a strong influence in the revival of interest in mathematics in his time. He was, moreover, a vital source of inspiration to mathematicians more than thirteen hundred years later when his surviving writings were a stimulating force upon the greatest scholars of the seventeenth century. Among them were Descartes, Fermat, Pascal and Desargues.

The works ascribed to Pappus by the Greek lexicographer, Suidas, indicate a breadth of learning on the part of the ancient author found only among the greatest of the ancient writers. In addition to Pappus's *Mathematical Collection*, originally in eight books, Suidas's list of works by Pappus contains *A Commentary on Ptolemy's Almagest*, *A Description of the Rivers of Libya*, *A Treatise on Military Engines* and *Commentaries on Aristarchus of Samos*, the last mentioned being concerned with the magnitudes and distances of the sun and moon. Of these, only the *Mathematical Collection* has been preserved. Although all of Book I and fragments of the other books of the *Collection* have been lost, the remaining parts of this extraordinary work form one of our richest sources of information about ancient mathematicians and the ancient science of mathematics.

The *Collection* became known in Europe through a Latin version and commentaries on it by Commandinus, published in 1588. In the *Collection*, Pappus gives a brief account of the contents of numerous older mathematical works which were held in high esteem in Alexandria. His famous *Treasury of Analysis* (*Mathematical Collection*, Book VII) contains a list of works written by Euclid, Apollonius, Aristaeus and Eratosthenes, which Pappus advises as most suitable for an advanced course of study in the method of analysis and synthesis. The procedure

employed by Pappus in discussing individual problems has been of inestimable historical value. He first presents the solutions given by previous authors, then his own demonstrations, and finally various applications. The *Collection* also contains the celebrated Problem of Pappus taken by Descartes for the third appendix (La Géométrie) to his *Discours de la Méthode*. As an illustration of the power of his *Méthode*, Descartes solved the problem proposed by Pappus and in the course of the solution, he was led to the invention of analytic geometry. Pappus's efforts could not prevent the deterioration in the study of mathematics in the centuries immediately following his death. However, mathematicians of a much later date eagerly studied all his available writings, and building upon his teachings, opened the way for the development of modern mathematics.

PAPPUS'S COLLECTION

From Selections Illustrating the History of Greek Mathematics
with an English translation by Ivor Thomas

(*f*) EXTENSION OF PYTHAGORAS'S THEOREM

Pappus, *Collection* iv.1.1, ed. Hultsch 176. 9–178. 13

If ABI' be a triangle, and on AB, $B\Gamma$ there be described any parallelograms $AB\Delta E$, $B\Gamma ZH$, and ΔE, ZH be produced to Θ, and ΘB be joined, then the parallelograms $AB\Delta E$, $B\Gamma ZH$ are together equal to the parallelogram contained by $A\Gamma$, ΘB in an angle which is equal to the sum of the angle $BA\Gamma$, $A\Theta B$.

For let ΘB be produced to K, and through A, Γ let $A\Lambda$, ΓM be drawn parallel to ΘK, and let ΛM be joined. Since $A\Lambda\Theta B$ is a parallelogram, $A\Lambda$, ΘB are equal and parallel. Similarly $M\Gamma$, ΘB are equal and parallel, so that ΛA, $M\Gamma$ are equal and parallel. And therefore ΛM, $A\Gamma$ are equal and parallel; therefore $A\Lambda M\Gamma$ is a parallelogram in the angle $\Lambda A\Gamma$, that is an angle equal to the sum of the angles $BA\Gamma$ and $\Delta\Theta B$; for the angle $\Delta\Theta B$ = angle ΛAB. And since the parallelogram ΔABE is equal to the parallelogram $\Lambda AB\Theta$ (for they are upon the same base AB and in the same parallels AB, $\Delta\Theta$), while $\Lambda AB\Theta = \Lambda AKN$ (for they are upon the

same base ΛA and in the same parallels ΛA, ΘK), therefore $A\Delta EB = \Lambda AKN$. By the same reasoning $BHZ\Gamma = NK\Gamma M$; therefore the parallelograms ΔABE, $BHZ\Gamma$ are together equal to $\Lambda A\Gamma M$, that is, to the parallelogram contained by $A\Gamma$, ΘB in the

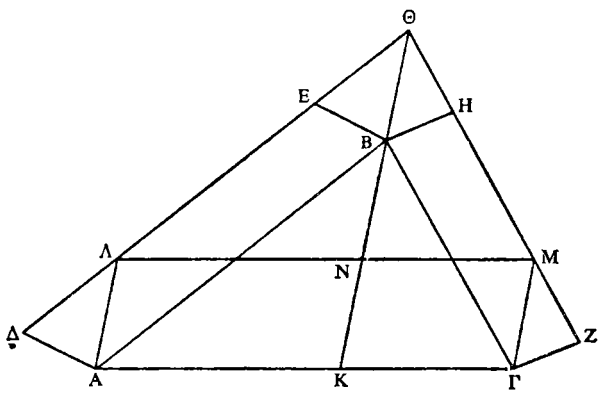


Fig. 29.1.

angle $\Lambda A\Gamma$, which is equal to the sum of the angles $BA\Gamma$, $B\Theta\Delta$. And this is much more general than the theorem proved in the *Elements* about the squares on right-angled triangles.[1]

(*g*) CIRCLES INSCRIBED IN THE *ΑΡΒΗΑΟΣ*

Ibid., iv. 14. 19, ed. Hultsch 208. 9–21.

There is found in certain [books] an ancient proposition to this effect: Let $AB\Gamma$, $A\Delta E$, $EZ\Gamma$ be supposed to be three semicircles touching each other, and in the space between their circumferences, which is called the 'leather-worker's knife', let there be inscribed any number whatever of circles touching both the

1. Eucl. i, 47, *v.* vol. i. pp. 178–85. In the case taken by Pappus, the first two parallelograms are drawn outwards and the third, equal to their sum, is drawn inwards. If the areas of parallelograms drawn outwards be regarded as of opposite sign to the areas of those drawn inwards, the theorem may be still further generalized, for the algebraic sum of the three parallelograms is equal to zero.

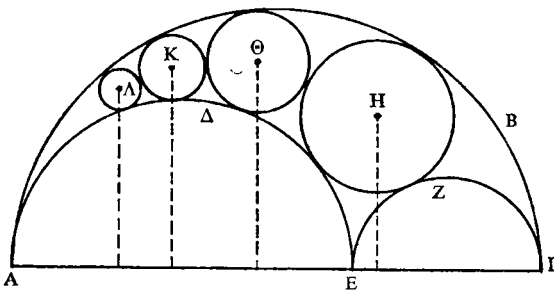


Fig. 29.2.

semicircles and one another, as those about the centres H, Θ, K, Λ; to prove that the perpendicular from the centre H to $A\Gamma$ is equal to the diameter of the circle about H, the perpendicular from Θ is double of the diameter of the circle about Θ, the perpendicular from K is triple, and the [remaining] perpendiculars in order are so many times the diameters of the proper circles according to the numbers in a series increasing by unity, the inscription of the circles proceeding without limit.[1]

(*i*) ISOPERIMETRIC FIGURES[2]

Ibid. v., Preface 1–3, ed. Hultsch 304. 5–308. 5

Though God has given to men, most excellent Megethion, the best and most perfect understanding of wisdom and mathematics,

1. Three propositions [about the 'leather-worker's knife' are] in Archimedes' *Liber Assumptorum*, which has survived in Arabic. They are included as particular cases in Pappus's exposition, which is unfortunately too long for reproduction here. Professor D'Arcy W. Thompson (*The Classical Review*, LVI (1942), pp. 75–6) gives reasons for thinking that this was a saddler's knife rather than a shoemaker's knife, as usually translated.

2. The whole of Book V in Pappus's *Collection* is devoted to isoperimetry. The first section follows closely the exposition of Zenodorus as given by Theon (*v. supra*, pp. 386–95), except that Pappus includes the proposition that *of all circular segments having the same circumference the semicircle is the greatest*. The second section compares the volumes of solids whose surfaces are equal, and is followed by a digression, on the semi-regular solids discovered by Archimedes. After some propositions on the lines of

He has allotted a partial share to some of the unreasoning creatures as well. To men, as being endowed with reason, He granted that they should do everything in the light of reason and demonstration, but to the other unreasoning creatures He gave only this gift, that each of them should, in accordance with a certain natural forethought, obtain so much as is needful for supporting life. This instinct may be observed to exist in many other species of creatures, but it is specially marked among bees. Their good order and their obedience to the queens who rule in their commonwealths are truly admirable, but much more admirable still is their emulation, their cleanliness in the gathering of honey, and the forethought and domestic care they give to its protection. Believing themselves, no doubt, to be entrusted with the task of bringing from the gods to the more cultured part of mankind a share of ambrosia in this form, they do not think it proper to pour it carelessly into earth or wood or any other unseemly and irregular material, but, collecting the fairest parts of the sweetest flowers growing on the earth, from them they prepare for the reception of the honey the vessels called honeycombs, [with cells] all equal, similar and adjacent, and hexagonal in form.

That they have contrived this in accordance with a certain geometrical forethought we may thus infer. They would necessarily think that the figures must all be adjacent one to another and have their sides common, in order that nothing else might fall into the interstices and so defile their work. Now there are only three rectilineal figures which would satisfy the condition, I mean regular figures which are equilateral and equiangular, inasmuch as irregular figures would be displeasing to the bees. For equilateral triangles and squares and hexagons can lie adjacent to one another and have their sides in common without irregular interstices. For the space about the same point can be filled by six equilateral triangles and six angles, of which each is

Archimedes' *De sph. et cyl.*, Pappus finally proves that *of regular solids having equal surfaces, that is greatest which has most faces.*

The introduction, here cited, on the sagacity of bees is rightly praised by Heath (*H.G.M.* ii 389) as an example of the good style of the Greek mathematicians when freed from the restraints of technical language.

$\frac{2}{3}$ right angle, or by four squares and four right angles, or by three hexagons and three angles of a hexagon, of which each is $1\frac{1}{3}$ right angle. But three pentagons would not suffice to fill the space about the same point, and four would be more than sufficient; for three angles of the pentagon are less than four right angles (inasmuch as each angle is $1\frac{1}{5}$ right angle), and four angles are greater than four right angles. Nor can three heptagons be placed about the same point so as to have their sides adjacent to each other; for three angles of a heptagon are greater than four right angles (inasmuch as each is $1\frac{3}{7}$ right angle). And the same argument can be applied even more to polygons with a greater number of angles. There being, then, three figures capable by themselves of filling up the space around the same point, the triangle, the square and the hexagon, the bees in their wisdom chose for their work that which has the most angles, perceiving that it would hold more honey than either of the two others.

Bees, then, know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each. But we, claiming a greater share in wisdom than the bees, will investigate a somewhat wider problem, namely that, *of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest of them all is the circle having its perimeter equal to them.*

(*j*) APPARENT FORM OF A CIRCLE

Ibid. vi.[1] 48. 90–91; ed. Hultsch 580. 12–27.

Let $AB\Gamma$ be a circle with centre E, and from E let EZ be drawn perpendicular to the plane of the circle; I say that, if the eye be placed on EZ, the diameters of the circle appear equal.[2]

This is obvious; for all the straight lines falling from Z on the circumference of the circle are equal one to another and contain equal angles.

1. Most of Book VI is astronomical, covering the treatises in the *Little Astronomy*. The proposition here cited comes from a section on Euclid's *Optics*.

2. As they will do if they subtend an equal angle at the eye.

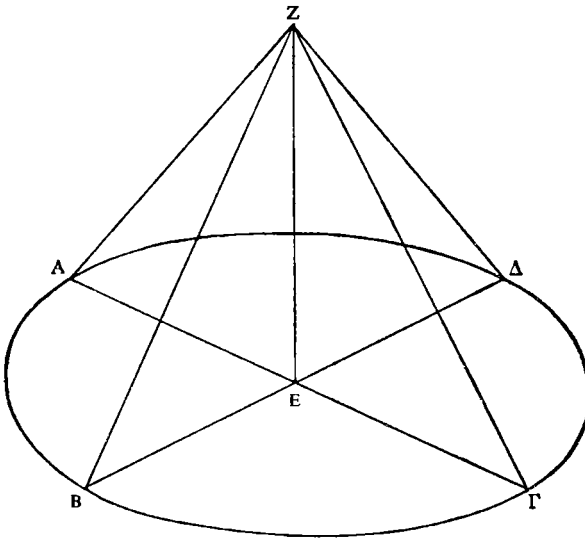


Fig. 29.3.

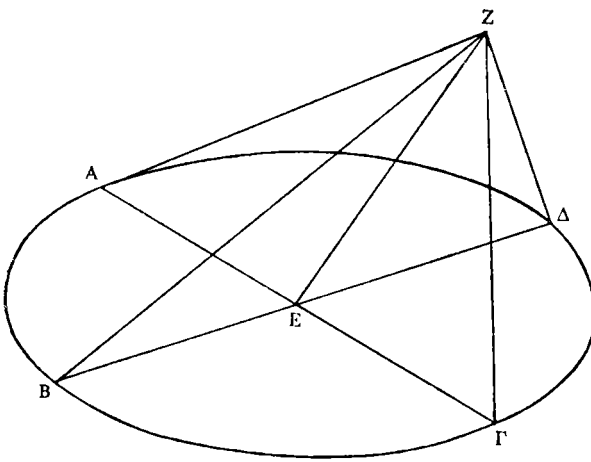


Fig. 29.4.

Now let EZ be not perpendicular to the plane of the circle, but equal to the radius of the circle; I say that, if the eye be at the point Z, in this case also the diameters appear equal.

For let two diameters $A\Gamma$, $B\Delta$ be drawn, and let ZA, ZB, $Z\Gamma$, $Z\Delta$ be joined. Since the three straight lines EA, $E\Gamma$, EZ are equal, therefore the angle $AZ\Gamma$ is right. And by the same reasoning the angle $BZ\Delta$ is right; therefore the diameters $A\Gamma$, $B\Delta$ appear equal. Similarly we may show that all are equal.

(*k*) THE 'TREASURY OF ANALYSIS'

Ibid. vii., Preface 1–3, ed. Hultsch 634. 3–636. 30

The so-called *Treasury of Analysis*, my dear Hermodorus, is, in short, a special body of doctrine furnished for the use of those who, after going through the usual elements, wish to obtain power to solve problems set to them involving curves, and for this purpose only is it useful. It is the work of three men, Euclid the writer of the *Elements*, Appollonius of Perga and Aristaeus the elder, and proceeds by the method of analysis and synthesis.

Now *analysis* is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a reverse solution.

But in *synthesis*, proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought; and this we call synthesis.

Now analysis is of two kinds, one, whose object is to seek the truth, being called *theoretical*, and the other, whose object is to find something set for finding, being called *problematical*. In the theoretical kind we suppose the subject of the inquiry to exist and

to be true, and then we pass through its consequences in order, as though they also were true and established by our hypothesis, to something which is admitted; then, if that which is admitted be true, that which is sought will also be true, and the proof will be the reverse of the analysis, but if we come upon something admitted to be false, that which is sought will also be false. In the problematical kind we suppose that which is set as already known, and then we pass through its consequences in order, as though they were true, up to something admitted; then, if what is admitted be possible and can be done, that is, if it be what the mathematicians call *given*, what was originally set will also be possible, and the proof will again be the reverse of the analysis, but if we come upon something admitted to be impossible, the problem will also be impossible.

So much for analysis and synthesis.

This is the order of the books in the aforesaid *Treasury of Analysis*. Euclid's *Data*, one book, Apollonius's *Cutting-off of a Ratio*, two books, *Cutting-off of an Area*, two books, *Determinate Section*, two books, *Contacts*, two books, Euclid's *Porisms*, three books, Apollonius's *Vergings*, two books, his *Plane Loci*, two books, *Conics*, eight books, Aristaeus's *Solid Loci*, five books, Euclid's *Surface Loci*, two books, Eratosthenes' *On Means*, two books. In all there are thirty-three books, whose contents as far as Apollonius's *Conics* I have set out for your examination, including not only the number of the propositions, the conditions of possibility and the cases dealt with in each book, but also the lemmas which are required; indeed, I believe that I have not omitted any inquiry arising in the study of these books.

(*l*) LOCUS WITH RESPECT TO FIVE OR SIX LINES[1]

If from any point straight lines be drawn to meet at given angles five straight lines given in position, and the ratio be given between the volume of the rectangular parallelepiped contained by three of them to the volume of the rectangular parallelepiped contained by the remaining two and a given straight line, the point

1. It was by reflection on this passage that Descartes evolved the system of co-ordinates described in his *Géométrie*.

will lie on a curve given in position. If there be six straight lines, and the ratio be given between the volume of the aforesaid solid formed by three of them to the volume of the solid formed by the remaining three, the point will again lie on a curve given in position. If there be more than six straight lines, it is no longer permissible to say 'if the ratio be given between some figure contained by four of them to some figure contained by the remainder', since no figure can be contained in more than three dimensions. It is true that some recent writers have agreed among themselves to use such expressions, but they have no clear meaning when they multiply the rectangle contained by these straight lines with the square on that or the rectangle contained by those. They might, however, have expressed such matters by means of the composition of ratios, and have given a general proof both for the aforesaid propositions and for further propositions after this manner: *If from any point straight lines be drawn to meet at given angles straight lines given in position, and there be given the ratio compounded of that which one straight line so drawn bears to another, that which a second bears to a second, that which a third bears to a third, and that which the fourth bears to a given straight line – if there be seven, or, if there be eight, that which the fourth bears to the fourth – the point will lie on a curve given in position;* and similarly, however many the straight lines be, and whether odd or even. Though, as I said, these propositions follow the locus on four lines, [geometers] have by no means solved them to the extent that the curve can be recognized.

(*m*) ANTICIPATION OF GULDIN'S THEOREM[1]

The men who study these matters are not of the same quality as the ancients and the best writers. Seeing that all geometers are occupied with the first principles of mathematics and the natural origin of the subject-matter of investigation, and being ashamed

1. Paul Guldin (1577–1643), or Guldinus, is generally credited with the discovery of the celebrated theorem here enunciated by Pappus. It may be stated: *If any plane figure revolve about an external axis in its plane, the volume of the solid figure so generated is equal to the product of the area of the figure and the distance travelled by the centre of gravity of the figure.* There is a corresponding theorem for the area.

to pursue such topics myself, I have proved propositions of much greater importance and utility . . . and in order not to make such a statement with empty hands, before leaving the argument I will give these enunciations to my readers. *Figures generated by a complete revolution of a plane figure about an axis are in a ratio compounded* (*a*) *of the ratio* [*of the areas*] *of the figures, and* (*b*) *of the ratio of the straight lines similarly drawn to the axes of rotation from the respective centres of gravity. Figures generated by incomplete revolutions are in a ratio compounded* (*a*) *of the ratio* [*of the areas*] *of the figures, and* (*b*) *of the ratio of the arcs described by the centres of gravity of the respective figures, the ratio of the arcs being itself compounded* (1) *of the ratio of the straight lines similarly drawn* [*from the respective centres of gravity to the axes of rotation*] *and* (2) *of the ratio of the angles contained about the axes of revolution by the extremities of these straight lines.* These propositions, which are practically one, include a large number of theorems of all sorts about curves, surfaces and solids, all of which are proved simultaneously by one demonstration, and include propositions never before proved as well as those already proved, such as those in the twelfth book of these elements.

30
PROCLUS
(410–85)

PROCLUS, DIADOCHUS PLATONICUS, Greek philosopher, mathematician and astronomer, was the foremost representative of the later Neo-Platonic school in Athens. At the age of thirty he became head of the Athenian Academy whose long, though broken history began with its founding by Plato early in the fourth century B.C. The name 'Diadochus Platonicus' was given to Proclus as a mark of his eminence not only because he was Plato's successor as head of the Academy but also because he epitomized many of the personal traits and ideals of his predecessor. Both Plato and Proclus were born of aristocratic and wealthy families, and each used his wealth unselfishly. Both possessed extraordinary intellectual powers; both were exceptionally gifted in physical strength and personal charm; both were motivated in their actions by high moral principles. The fame and influence of the Academy increased in Proclus's time through his forceful and untiring efforts in demonstrating the basic values of Plato's teachings. Proclus was deeply devoted to the old pagan religion.

Proclus was born in Byzantium in 410. Shortly after his birth his parents brought him to their native city of Xanthus in Lycia. He received his early education in Xanthus. Intended to follow his father's profession, the study of law, he was sent to Alexandria where he attended the Roman schools. At the same time he received instruction from the Greek masters, Leonas the rhetorician and Orion the grammarian. Philosophy and rhetoric proved to be more to his liking than Roman law, and, when Leonas left Alexandria on a mission to Byzantium, Proclus accompanied him on this voyage.

When he returned to Alexandria, Proclus turned to the study of mathematics and the doctrines of Aristotle. Later, upon hearing that Athens was the seat of the most brilliant teaching in philosophy, he left Alexandria for Athens. He was not yet twenty when Syrianus brought him to Plutarch, then head of the Academy in Athens. The ageing Plutarch, profoundly moved by the manners and the scholarly

attainments of young Proclus, accepted him as a pupil, and took him into his home as his own son. Proclus was initiated by Plutarch into the art of writing commentaries on the great philosophers. After Plutarch's death, Proclus became the companion of Syrianus under whom he studied the philosophy of Plato and by whom he was trained to assume the responsibilities of the head of the Academy. At the age of twenty-eight, Proclus had already completed a great number of detailed treatises and commentaries. To his skill in poetry and philosophy were added a wide knowledge of the sciences. Some called him the 'prince of all sciences'.

While Proclus's main interest was in the exposition of Plato's writings, he encouraged the study of the sciences, giving particular attention to mathematics and astronomy. His writings include commentaries on scientific treatises which he believed were a proper part of the education of his pupils preparatory to their introduction to philosophy. Among these works are his *Commentaries on Book I of Euclid's Elements.* Writing for beginners in geometry, Proclus presented digests or the paraphrased contents of the works of the previous writers, adding his own comments and also his contributions to the subject (as in the case of his attempt to prove Euclid's parallel postulate). One of the two principal sources of our information about the origins of Greek mathematics, this work is of extreme historical interest. Many of the ancient treatises which were available to Proclus and which were described by him in the *Commentaries* are now lost. His recitals of their texts are in some instances invaluable evidence corroborating other sources. There are also some texts cited by him, which would be completely unknown to us but for his inclusion of them in his *Commentaries.*

PROCLUS'S SUMMARY[1]

From Selections Illustrating the History of Greek Mathematics
with an English Translation by Ivor Thomas

Since it behoves us to examine the beginnings both of the arts and of the sciences with reference to the present cycle [of the universe], we say that according to most accounts geometry was first dis-

1. This commentary is one of the two main sources for the history of Greek geometry, the other being the *Collection* of Pappus.

covered among the Egyptians,[1] taking its origin from the measurement of areas. For they found it necessary by reason of the rising of the Nile, which wiped out everybody's proper boundaries. Nor is there anything surprising in that the discovery both of this and of the other sciences should have its origin in a practical need, since everything which is in process of becoming progresses from the imperfect to the perfect. Thus the transition from perception to reasoning and from reasoning to understanding is natural. Just as exact knowledge of numbers received its origin among the Phoenicians by reason of trade and contracts, even so geometry was discovered among the Egyptians for the aforesaid reason.

Thales[2] was the first to go to Egypt and bring back to Greece this study; he himself discovered many propositions, and disclosed the underlying principles of many others to his successors, in some cases his method being more general, in others more

1. The Egyptian origin of geometry is taught by Herodotus ii. 109, where it is asserted that Sesostris (Ramses II, *c.* 1300 B.C.) divided the land among the Egyptians in equal rectangular plots, on which an annual tax was levied; when therefore the river swept away a portion of a plot, the owner applied for a reduction of tax, and surveyors had to be sent down to report. In this he saw the origin of geometry, and this story may be the source of Proclus's account, as also of the similar accounts in Heron, *Geometrica* 2, ed. Heiberg 176, 1–13, Diodorus Siculus i. 69, 81 and Strabo xvii. c. 3. Aristotle also finds the origin of mathematics among the Egyptians, but in the existence of a leisured class of priests, not in a practical need (*Metaphysica* A 1, 981 b 23).

2. Thales (*c.* 624–547 B.C.), one of the 'Seven Wise Men' of ancient Greece, is universally acknowledged as the founder of Greek geometry, astronomy and philosophy. His greatest fame in antiquity rested on his prediction of the total eclipse of the sun of 28 May, 585 B.C., which led to the cessation of hostilities between the Medes and Lydians and a lasting peace (Herodotus i. 74); what Thales probably did was to predict the *year* in which the eclipse would take place, an achievement by no means beyond the astronomical powers of the age. Thales was noted for his political sense. He urged the separate states of Ionia, threatened by the encroachment of the Lydians, to form a federation with a capital at Teos; and his successful dissuasion of his fellow-Milesians from accepting the overtures of Croesus, king of the Lydians, may have had an influence on the favourable terms later granted to Miletus by Cyrus, king of the Persians, though the main reason for this preferential treatment was probably commercial. In philosophy Thales taught that the all is water.

empirical. After him Ameristus,[1] the brother of the poet Stesichorus, is mentioned as having touched the study of geometry, and Hippias of Elis[2] spoke of him as having acquired a reputation for geometry. After these Pythagoras[3] transformed this study into the form of a liberal education, examining its principles from the beginning and tracking down the theorems immaterially and intellectually; he it was who discovered the theory of proportionals[4] and the construction of the cosmic figures. After him Anaxagoras of Clazomenae[5] touched many questions affecting geometry, and so did Oenopides of Chios,[6] being a little younger than Anaxagoras, both of whom Plato mentioned in the *Rivals* as having acquired a reputation for mathematics.

After them Hippocrates of Chios,[7] who discovered the quadra-

1. The name is uncertain. Stesichorus, the lyric poet, flourished *c.* 611 B.C.

2. The well-known Sophist, born about 460 B.C., whose various accomplishments are described in Plato's *Hippias Minor*. He claimed to have gone once to the Olympic Games with everything that he wore made by himself, as well as all kinds of works in prose and verse by his own composition. His system of mnemonics enabled him to remember any string of fifty names which he had heard once. The unmathematical Spartans, however, could not appreciate his genius, and from them he could get no fees. His chief mathematical discovery was the curve known as the quadratrix, which could be used for trisecting an angle or squaring the circle.

3. The life of Pythagoras is shrouded in mystery. He was probably born in Samos about 582 B.C. and migrated about 529 B.C. to Crotona, the Dorian colony in southern Italy, where a semi-religious brotherhood sprang up round him. This brotherhood was subjected to severe persecution in the fifth century B.C., and the Pythagoreans then took their doctrines into Greece proper.

4. Friedlein's reading is, 'irrationals', but there is grave difficulty in believing that Pythagoras could have developed a theory of irrationals; in fact, a Pythagorean is said to have been drowned at sea for his impiety in disclosing the existence of irrationals.

5. *c.* 500–428 B.C. Clazomenae was a town near Smyrna. All we know about the mathematics of Anaxagoras is that he wrote on the squaring of the circle while in prison and may have written a book on perspective (Vitruvius, *De architectura* vii. praef. 11).

6. Oenopides was primarily an astronomer, and Eudemus is believed to have credited him with the discovery of the obliquity of the ecliptic and the period of the Great Year (Theon of Smyrna, ed. Hiller 198. 14–16). In mathematics Proclus attributed to him the discovery of Eucl. i. 12 and i. 23.

7. Hippocrates was in Athens from about 450 to 430 B.C.

ture of the lune, and Theodorus of Cyrene[1] became distinguished in geometry. For Hippocrates is the first of those mentioned as having compiled *elements*.[2] Plato, who came after them, made the other branches of mathematics as well as geometry take a very great step forward by his zeal for them; and it is obvious how he filled his writings with mathematical arguments and everywhere stirred up admiration for mathematics in those who took up philosophy. At this time also lived Leodamas of Thasos[3] and Archytas of Taras and Theaetetus of Athens, by whom the theorems were increased and an advance was made towards a more scientific grouping.

Younger than Leodamas were Neoclides and his pupil Leon, who added many things to those known before them, so that Leon was able to make a collection of the *elements* in which he was more careful in respect both of the number and of the utility of the things proved; he also discovered *diorismi*, showing when the problem investigated can be solved and when not. Eudoxus of Cnidos, a little younger than Leon and an associate of Plato's school, was the first to increase the number of the so-called general theorems; to the three proportions he added another three, and increased the number of theorems about the section, which had their origin with Plato, applying the method of analysis to them. Amyclas of Heraclea,[4] one of the friends of Plato, and Menaechmus,[5] a pupil of Eudoxus who had associated

1. Our chief knowledge of Theodorus comes from the *Theaetetus* of Plato, whose mathematical teacher he is said to have been (Diog. Laert. ii. 103).

2. Proclus (in Eucl. i., ed. Friedlein 72 seq.) explains that the *elements* in geometry are leading theorems having to those which follow the relation of an all-pervading principle; he compares them with the letters of the alphabet in relation to language; and they have, indeed, the same name in Greek.

3. All we know about him is that Plato is said to have explained or communicated to him the method of analysis (Diog. Laert. iii. 24, Procl. in Eucl. i., ed. Friedlein 211, 19–23).

4. The correct spelling appears to be Amyntas, though Diogenes Laertius (iii. 46) speaks of Amyclas of Heraclea as a pupil of Plato and in another place (ix. 40) says that a certain Pythagorean Amyclas dissuaded Plato from burning the works of Democritus. Heraclea was in Pontus.

5. He discovered the conic sections.

with Plato, and his brother Dinostratus[1] made the whole of geometry still more perfect. Theudius of Magnesia seemed to excel both in mathematics and in the rest of philosophy; for he made an admirable arrangement of *elements* and made many particular propositions more general. Again, Athenaeus of Cyzicus, who lived about those times, became famous in other branches of mathematics but mostly in geometry. They spent their time together in the Academy, conducting their investigations in common. Hermotimus of Colophon advanced farther the investigations begun by Eudoxus and Theaetetus; he discovered many propositions in the *Elements* and compiled some portion of the theory of loci. Philippus of Medma,[2] a disciple of Plato and by him diverted to mathematics, not only made his investigations according to Plato's directions but set himself to do such things as he thought would fit in with the philosophy of Plato.

Those who have compiled histories carry the development of this science up to this point. Not much younger than these is Euclid, who put together the *Elements*, arranging in order many of Eudoxus's theorems, perfecting many of Theaetetus's, and also bringing to irrefutable demonstration the things which had been only loosely proved by his predecessors. This man lived in the time of the first Ptolemy; for Archimedes, who came immediately after the first Ptolemy, makes mention of Euclid; and further they say that Ptolemy once asked him if there was in geometry a way shorter than that of the elements; he replied that there was no royal road to geometry.[3] He is therefore younger than the

1. He applied the quadratrix (probably discovered by Hippias) to the squaring of the circle.

2. Almost certainly the same as Philippus of Opus, who is said to have revised and published the *Laws* of Plato and (wrongly) to have written the *Epinomis*. Suidas notes a number of astronomical and mathematical works by him.

3. Not much more is known about the life of Euclid than is contained in this passage (see Heath, *The Thirteen Books of Euclid's Elements*, vol. i., pp. 1–6 and *H.G.M.* i. 354 7). The summary of Euclid's achievement in the *Elements* is a very fair one, agreeing with the considered judgement of Heath (*H.G.M.* i. 217): 'There is therefore probably little in the whole compass of the *Elements* of Euclid, except the new theory of proportion due to Eudoxus and its consequences, which was not in substance included in the recognized content of geometry and arithmetic by Plato's time, al-

pupils of Plato, but older than Eratosthenes and Archimedes. For these men were contemporaries, as Eratosthenes[1] somewhere says. In his aim he was a Platonist, being in sympathy with this philosophy, whence it comes that he made the end of the whole *Elements* the construction of the so-called Platonic figures.[2] There are many other mathematical writings by this man, wonderful in their accuracy and replete with scientific investigations. Such are the *Optics* and *Catoptrics*, and the *Elements of Music*, and again the book *On Divisions*.[3] He deserves admiration pre-eminently in

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though the form and arrangement of the subject-matter and the method employed in particular cases were different from what we find in Euclid' (cf. *H.G.M.* i. 357). As Plato died in 347 B.C., and Archimedes was born in 287 B.C., Euclid must have flourished about 300 B.C.; Ptolemy I reigned from 306 to 283 B.C. Had not the confusion been common in the Middle Ages, it would scarcely be necessary to point out that this Euclid is to be distinguished from Euclid of Megara, the philosopher, who lived about 400 B.C. A story about there being no royal road to geometry is also told of Menaechmus and Alexander (Stobaeus, *Ecl.* ii. 31, ed. Wachsmuth 115).

1. Eratosthenes was born about 284 B.C. His ability in many branches of knowledge, but failure to achieve the highest place in any, won for him the nicknames 'Beta' and 'Pentathlos'. He became tutor to Philopator, son of Ptolemy Euergetes and librarian at Alexandria. He wrote a book *Platonicus* and another *On Means* (both lost). His greatest achievement was his measurement of the circumference of the earth to a surprising degree of exactitude. (See Heath, *H.G.M.* i. 106–8, *Greek Astronomy*, pp. 109–12.)

2. It is true that the final book of the *Elements*, as written by Euclid, dealt with the construction of the cosmic, or Platonic, figures, but the whole work was certainly not designed with a view to their construction. Euclid, however, may quite well have been a Platonist.

3. Euclid's *Optics* survives and is available in the Teubner text in two recensions, one probably Euclid's own, the other by Theon of Alexandria. It is possible that Proclus has attributed to Euclid a treatise on *Catoptrics* (Mirrors) which was really Theon's; a treatise by Euclid on this subject is not otherwise known. Two musical treatises attributed to Euclid are extant, the *Sectio Canonis* and the *Introductio Harmonica*; the latter, however, is definitely by Cleonides, a pupil of Aristoxenus, and it is not certain that the former is Euclid's own. The book *On Divisions (of Figures)* has survived in an Arabic text discovered by Woepcke at Paris and published in 1851; see R. C. Archibald, *Euclid's Book on Division of Figures with a restoration based on Woepcke's text and the Practica Geometriae of Leonardo Pisano* (Cambridge 1915). A Latin translation (probably by Gherard of Cremona, 1114–87) from the Arabic was known in the Middle Ages, but the Arabic cannot have been a direct translation from Euclid's Greek. The general

the compilation of his *Elements of Geometry* on account of the order and of the selection both of the theorems and of the problems made with a view to the elements. For he included not everything which he could have said, but only such things as he could set down as elements. And he used all the various forms of syllogisms, some getting their plausibility from the first principles, some setting out from demonstrative proofs, all being irrefutable and accurate and in harmony with science. In addition to these he used all the dialectical methods, the *divisional* in the discovery of figures, the *definitive* in the existential arguments, the *demonstrative* in the passages from first principles to the things sought, and the *analytic* in the converse process from the things sought to the first principles. And the various species of conversions,[1] both of the simpler (propositions) and of the more complex, are in this treatise accurately set forth and skilfully investigated, what wholes can be converted with wholes, what wholes with parts and conversely, and what as parts with parts. Again, mention must be made of the continuity of the proofs, the disposition and arrangement of the things which precede and those which follow, and the power with which he treats each detail.

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character of the treatise is indicated by Procl. in *Eucl.* i., ed. Friedlein 144, 22–6, as the division of figures into like and unlike figures.

1. Geometrical conversion is to be distinguished from logical conversion, as described by Aristotle, *Cat.* xii. 6 and elsewhere. An analysis of the conversion of geometrical propositions is given by Proclus (in *Eucl.* i., ed. Friedlein, 252. 5 seq.). In the leading form of conversion, also called conversion *par excellence*, the conversion is simple, the hypothesis and conclusion of one theorem becoming the conclusion and hypothesis of the converse theorem. The other form of conversion is more complex, being that where several hypotheses are combined into a single enunciation so as to lead to a single conclusion. In the converse proposition the conclusion of the original proposition is combined with the hypotheses of the original proposition, less one, so as to lead to the omitted hypothesis as the new conclusion. An example of the first species of conversion is Euclid i. 6, which is the converse of Euclid i. 5, and Heath's notes thereon are most valuable (*The Thirteen Books of Euclid's Elements*, vol. i, pp. 256–7); an example of partial conversion is given by Euclid i. 8, which is a converse to i. 4.

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